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Holomorphic Morse inequalities for orbifolds

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Abstract We prove that Demailly's holomorphic Morse inequalities hold true for complex orbifolds by using a heat kernel method. Then we introduce the class of Moishezon orbifolds and as an application of our inequalties, we give a geometric criterion for a compact connected orbifold to be a Moishezon orbifolds, thus generalizing Siu's and Demailly's answers to the Grauert–Riemenschneider conjecture to the orbifold case.

Keywords Holomorphic Morse inequalities · Orbifold · Dolbeault cohomology · Heat kernel · High tensor powers of line bundles

Mathematics Subject Classification 51M16 · 57R18 · 32L05 · 32L10 · 32W30

1 Introduction

Morse Theory investigates the topological information carried by Morse functions on a manifold and in particular their critical points. Let f be a Morse function on a compact manifold of real dimension n. Let m_j $(0 \leq j \leq n)$ be the number of critical points of f of Morse index j, and let b_j be the Betti numbers of the manifold. Then the strong Morse inequalities state that for $0 \leq q \leq n$,

$$\sum_{j=0}^{q}(-1)^{q-j}b_j \leq \sum_{j=0}^{q}(-1)^{q-j}m_j, \tag{1}$$

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with equality if $q = n$. From (1), we get the weak Morse inequalities:

$$b_j \leq m_j \quad \text{for} \quad 0 \leq j \leq n. \tag{2}$$

In his seminal paper [26], Witten gave an analytic proof of the Morse inequalities by analyzing the spectrum of the Schrödinger operator $\Delta_t = \Delta + t^2|df|^2 + tV$, where $t > 0$ is a real parameter and V an operator of order 0. For $t \to +\infty$, Witten shows that the spectrum of Δ_t approaches in some sense the spectrum of a sum of harmonic oscillators attached to the critical points of f.

In [10], Demailly established analogous asymptotic Morse inequalities for the Dolbeault cohomology associated with high tensor powers $L^p := L^{\otimes p}$ of a (smooth) holomorphic Hermitian line bundle (L, h^L) over a (smooth) compact complex manifold (M, J) of dimension n. The inequalities of Demailly give asymptotic bounds on the Morse sums of the Betti numbers of $\bar{\partial}$ on L^p in terms of certain integrals of the Chern curvature R^L of (L, h^L). More precisely, we define $\dot{R}^L \in \mathrm{End}(T^{(1,0)}M)$ by $g^{TM}(\dot{R}^L u, \overline{v}) = R^L(u, \overline{v})$ for $u, v \in T^{(1,0)}M$, where g^{TM} is any J-invariant Riemannian metric on TM. We denote by $M(q)$ the set of points where $\dot{R}^L$ is non-degenerate and exactly q negative eigenvalues, and we set $M(\leq q) = \cup_{j \leq q} M(j)$. Let $n = \dim_{\mathbb{C}} M$, then we have for $0 \leq q \leq n$

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p) \leq \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n), \tag{3}$$

with equality if $q = n$. Here $H^j(M, L^p)$ denotes the Dolbeault cohomology in bidegree $(0, j)$, which is also the j-th group of cohomology of the sheaf of holomorphic sections of L^p. Note that $M(q)$ and $M(\leq q)$ are open subsets of M and do not depend on the metric g^{TM}.

These inequalities have found numerous applications. In particular, Demailly used them in [10] to find new geometric characterizations of Moishezon spaces, which improve Siu's solution in [23,24] of the Grauert–Riemenschneider conjecture [14]. Another notable application of the holomorphic Morse inequalities is the proof of the effective Matsusaka theorem by Siu [12,25]. Recently, Demailly used these inequalities in [13] to prove a significant step of a generalized version of the Green–Griffiths–Lang conjecture.

To prove these inequalities, the key remark of Demailly was that in the formula for the Kodaira Laplacian $\square_p$ associated with L^p, the metric of L plays formally the role of the Morse function in the paper Witten [26], and that the parameter p plays the role of the parameter t. Then the Hessian of the Morse function becomes the curvature of the bundle. The proof of Demailly was based on the study of the semi-classical behavior as $p \to +\infty$ of the spectral counting functions of $\square_p$. Subsequently, Bismut gave an other proof of the holomorphic Morse inequalities in [3] by adapting his heat kernel proof of the Morse inequalities [2]. The key point is that we can compare the left hand side of (3) with the alternate trace of the heat kernel acting on forms of degree $\leq q$ (see [3, Theorem 1.4]):

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p) \leq \sum_{j=0}^{q} (-1)^{q-j} \operatorname{Tr} \left[e^{-\frac{u}{p} \square_p} |_{\Omega^{0,j}(M, L^p)} \right], \tag{4}$$

with equality if $q = n$. Then, Bismut obtained the holomorphic Morse inequalities by showing the convergence of the heat kernel thanks to probability theory. Demailly [11] and Bouche [6] gave an analytic approach of this result. In [20], Ma and Marinescu gave a new proof of this convergence, replacing the probabilistic arguments of Bismut [3] by arguments inspired by the analytic localization techniques of Bismut–Lebeau [5, Chapter 11].

When the bundle L is positive, (3) is a consequence of the Hirzebruch–Riemann–Roch theorem and of the Kodaira vanishing theorem, and reduces to

$$\dim H^0(M, L^p) = \frac{p^n}{n!} \int_M \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n). \tag{5}$$

In this case, a local estimate can be obtained by the study of the asymptotic of the Bergman kernel (the kernel of the orthogonal projection from $\mathscr{C}^\infty(M, L^p)$ onto $H^0(M, L^p)$) when $p \to +\infty$. We refer to [20] and the reference therein for the study of the Bergman kernel.

It is a natural question to know whether we can prove a version of Demailly's holomorphic Morse inequalities when M is a complex orbifold and L is an orbifold bundle. Applying the result of [15,16], one can prove that such inequalities hold if M is the quotient of a CR manifold by a transversal CR $\mathbb{S}^1$-action. In this paper, we prove that Demailly's inequalities hold for high tensor power of an orbifold line bundle, twisted by another orbifold bundle, on a general compact complex orbifolds. We also introduce the class of Moishezon orbifolds and as an application of our inequalties, we give a geometric criterion for a compact connected orbifold to be a Moishezon orbifold, thus generalizing the above-mentioned Siu's and Demailly's answers to the Grauert–Riemenschneider conjecture to the orbifold case.

We now give more details about our results.

Let M be a compact complex orbifold of dimension n. We denote the complex structure of M by J. Let (L, h^L) a Hermitian holomorphic orbifold line bundle on M and let (E, h^E) be a Hermitian holomorphic orbifold vector bundle on M. As we will see in Sect. 2.3, we may assume without loss of generality that L and E are proper. We denote by R^L the Chern curvature of (L, h^L). We refer to Sect. 2 for the background concerning orbifolds, but for this introduction, let us just say that every object on an orbifold can be seen locally as being the quotient of an object on a non-singular manifold which is invariant by a finite group, and that we keep the same notation for both these objects.

We define $\dot{R}^L$, $M(q)$ and $M(\leq q)$ exactly as in the non-singular case above.

Let $H^\bullet(M, L^p \otimes E)$ be the orbifold Dolbeault cohomology. Then our holomorphic Morse inequalities for orbifolds have the same statement as the regular ones:

Theorem 1 *As $p \to +\infty$, the following strong Morse inequalities hold for $q \in \{1, \ldots, n\}$*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E) \leq \mathrm{rk}(E) \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n), \tag{6}$$

with equality for $q = n$.

In particular, we get the weak Morse inequalities

$$\dim H^q(M, L^p \otimes E) \leq \mathrm{rk}(E) \frac{p^n}{n!} \int_{M(q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n). \tag{7}$$

The integrals over an orbifold appearing here are defined by (18).

Of course, if the orbifold M is a non-singular manifold, and if the orbifold bundles L and E are usual bundles, then Theorem 1 coincides with the inequalities of Demailly.

Let us now give the main steps our proof. We draw our inspiration from the heat kernel method of [3] (see also [20, Sections. 1.6 and 1.7]), but the main difficulty when compared to this paper is that the singularities of M make it impossible to have an uniform asymptotics for the heat kernel (see Remark 3), and we thus have to work further near the singularities.

In particular we have to use the off-diagonal development of the heat kernel proved by Dai–Liu–Ma [9].

Note that in the case where L is positive, similar difficulties arise in the study of the Bergman kernel, and the asymptotic results concerning the heat kernel given below have parallel results for the Bergman kernel on orbifolds [20] (see also [9]).

Let g^{TM} be a Riemannian metric on TM which is compatible with J. Let

$$\square_p := \bar{\partial}^{L^p\otimes E}\bar{\partial}^{L^p\otimes E,*} + \bar{\partial}^{L^p\otimes E,*}\bar{\partial}^{L^p\otimes E} \tag{8}$$

be the Kodaira Laplacian acting on $\Omega^{0,\bullet}(M, L^p \otimes E)$ associated with $L^p \otimes E$, h^L, h^E and g^{TM} (see the beginning of Sect. 3 for details). Note that the operator $\square_p$ preserves the $\mathbb{Z}$-grading. We denote by $\mathrm{Tr}_q[e^{-\frac{u}{p}\square_p}]$ the trace of $e^{-\frac{u}{p}\square_p}$ acting on $\Omega^{0,q}(M, L^p \otimes E)$. We then have an analogue of (4):

Theorem 2 *For any $u > 0$, $p \in \mathbb{N}^*$ and $0 \leq q \leq n$, we have*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M, L^p \otimes E) \leq \sum_{j=0}^{q}(-1)^{q-j}\,\mathrm{Tr}_j\left[e^{-\frac{u}{p}\square_p}\right], \tag{9}$$

with equality for $q = n$.

From this result, we will prove our inequalities (6) by studying the asymptotics of the heat kernel. We begin by its behavior away from the singularities of M.

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$ (with the metric induced by g^{TM}) with dual frame $\{w^j\}_{j=1}^n$. Set

$$\omega_d = -\sum_{k,\ell} R^L(w_k, \overline{w}_\ell)\overline{w}^\ell \wedge i_{\overline{w}_k} \in \mathrm{End}(\Lambda^{0,\bullet}(T^*M)). \tag{10}$$

For compactness, in the sequel we will write

$$\mathcal{L}im_u(x) = \frac{1}{(2\pi)^n}\frac{\det(\dot{R}^L_x)e^{u\omega_{d,x}}}{\det\left(1 - \exp(-u\dot{R}^L_x)\right)} \otimes \mathrm{Id}_{E_x}, \tag{11}$$

with the convention that if an eigenvalue of $\dot{R}^L_x$ is zero, then its contribution to the term $\frac{\det(\dot{R}^L_x)}{\det\left(1-\exp(-u\dot{R}^L_x)\right)}$ is $\frac{1}{u}$.

Let M_{reg} be the regular part of M (see Sect. 2) and let $M_{sing} = M \setminus M_{reg}$ be the singular part of M.

Theorem 3 *For $K \subset M_{reg}$ compact, $u > 0$ and $\ell \in \mathbb{N}$, there exists $C > 0$ such that for any $x \in K$, we have as $p \to +\infty$*

$$\left| p^{-n}e^{-\frac{u}{p}\square_p}(x, x) - \mathcal{L}im_u(x)\right|_{\mathscr{C}^\ell} \leq Cp^{-1/2}. \tag{12}$$

Here, $|\cdot|_{\mathscr{C}^\ell}$ denotes the $\mathscr{C}^\ell$-norm.

Remark 1 In fact, the bound in the right hand side in Theorem 3 can be improved to p^{-1} using the same reasoning as in [20, Section 4.2.4] (see also [9]). However, we will not need this improvement and we leave it to the reader.

We now turn to the asymptotic behavior of the heat kernel near the singularities. This is the main technical innovation of this paper.

Let ∇^L and ∇^E be the Chern connections of (L, h^L) and (E, h^E), i.e., the unique connections preserving both the holomorphic and Hermitian structures.

Let $x_0 \in M_{sing}$, from Sect. 2 (and in particular Lemma 1), we know that we can identify an open neighborhood U_{x_0} of x_0 to $\widetilde{U}_{x_0}/G_{x_0}$, where $\widetilde{U}_{x_0} \subset \mathbb{C}^n$ is an open neighborhood of 0 on which the finite group G_{x_0} acts linearly and effectively. In this chart, x_0 correspond to the class [0]. Then the metric g^{TM} induces a G_{x_0}-invariant metric on $\widetilde{U}_{x_0}$.

Let $\widetilde{U}_{x_0}^g$ be the fixed point-set of $g \in G_{x_0}$, and let $\widetilde{N}_{x_0,g}$ be the normal bundle of $\widetilde{U}_{x_0}^g$ in $\widetilde{U}_{x_0}$. For each $g \in G_{x_0}$, the exponential map $Y \in (\widetilde{N}_{x_0,g})_{\tilde{x}} \mapsto \exp_{\tilde{x}}^{\widetilde{U}_{x_0}}(Y)$ identifies a neighborhood of $\widetilde{U}_{x_0}^g$ to $\widetilde{W}_{x_0,g} = \{Y \in \widetilde{N}_{x_0,g} \,:\, |Y| \leq \varepsilon\}$. We identify $L|_{\widetilde{W}_{x_0,g}}$ and $E|_{\widetilde{W}_{x_0,g}}$ to $L|_{\widetilde{U}_{x_0}^g}$ and $E|_{\widetilde{U}_{x_0}^g}$ by using the parallel transport (with respect to ∇^L and ∇^E) along the above exponential map. Then the action of g on $L|_{\widetilde{W}_{x_0,g}}$ is the multiplication by $e^{i\theta_g}$, and θ_g is locally constant on $\widetilde{U}_{x_0}^g$. Likewise, the action of g on $E|_{\widetilde{W}_{x_0,g}}$ is given by $g^E \in \mathscr{C}^\infty(\widetilde{U}_{x_0}^g, \mathrm{End}(E))$, and g^E is parallel with respect to ∇^E.

If $\widetilde{Z} \in \widetilde{W}_{x_0,g}$, we write $\widetilde{Z} = (\widetilde{Z}_{1,g}, \widetilde{Z}_{2,g})$ with $\widetilde{Z}_{1,g} \in \widetilde{U}_{x_0}^g$ and $\widetilde{Z}_{2,g} \in \widetilde{N}_{x_0,g}$.

On $\widetilde{U}_{x_0}$, we have two metrics: the first is the G-invariant lift $\widetilde{g^{TM}}$ of $g^{TM}|_{U_{x_0}}$ and the second is the constant metric $(\widetilde{g^{TM}})_{\widetilde{Z}=0}$. Let $dv_{\widetilde{M}}$ and $dv_{\widetilde{TM}}$ be the associated volume form, and let $\widetilde{\kappa}$ be the smooth positive function defined by

$$dv_{\widetilde{M}}(\widetilde{Z}) = \widetilde{\kappa}(\widetilde{Z}) dv_{\widetilde{TM}}(\widetilde{Z}), \tag{13}$$

with $\widetilde{\kappa}(0) = 1$.

For $x \in U_{x_0}$, the G-invariant lift of $\dot{R}_x^L$ acts on $T^{1,0}\widetilde{U}_{x_0}$, and we extend it to $T\widetilde{U}_{x_0} \otimes \mathbb{C} = T^{1,0}\widetilde{U}_{x_0} \oplus T^{0,1}\widetilde{U}_{x_0}$ by setting $\dot{R}_x^L \overline{\widetilde{v}} = -\overline{\dot{R}_x^L \widetilde{v}}$. We then define

$$\mathcal{E}_{g,x}(u, \widetilde{Z}) = \exp\left\{ -\left\langle \frac{\dot{R}_x^L/2}{\mathrm{th}(u\dot{R}_x^L/2)} \widetilde{Z}, \widetilde{Z} \right\rangle + \left\langle \frac{\dot{R}_x^L/2}{\mathrm{sh}(u\dot{R}_x^L/2)} e^{u\dot{R}_x^L/2} g^{-1}\widetilde{Z}, \widetilde{Z} \right\rangle \right\}. \tag{14}$$

Here again, we use the convention that if an eigenvalue of $\dot{R}_x^L$ is zero, then the contribution of the associated eigenspace to $\mathcal{E}_{g,x}(u, \widetilde{Z})$ is of the form $(u, \widetilde{V}) \mapsto e^{-\frac{1}{2u}|g^{-1}\widetilde{V} - \widetilde{V}|^2}$.

Theorem 4 *On $\widetilde{U}_{x_0}$, for $u > 0$ and $\ell \in \mathbb{N}$, there exist $c, C > 0$ and $N \in \mathbb{N}$ such that for any $|\widetilde{Z}| < \varepsilon/2$, as $p \to +\infty$*

$$\begin{aligned}
&\Bigg| p^{-n} e^{-\frac{u}{p}\Box_p}(\widetilde{Z}, \widetilde{Z}) - \mathcal{L}im_u(\widetilde{Z}) \\
&\quad - \sum_{\substack{g \in G_{x_0} \\ g \neq 1}} e^{ip\theta_g} g^E(\widetilde{Z}_{1,g}) \kappa_{\widetilde{Z}_{1,g}}^{-1}(\widetilde{Z}_{2,g}) \mathcal{L}im_u(\widetilde{Z}_{1,g}) \mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \sqrt{p}\widetilde{Z}_{2,g}) \Bigg|_{\mathscr{C}^\ell} \\
&\leq Cp^{-1/2} + Cp^{\frac{\ell-1}{2}} (1 + \sqrt{p}\, d(Z, M_{sing}))^N e^{-cp\, d(Z, M_{sing})^2}.
\end{aligned} \tag{15}$$

Remark 2 The term $\mathcal{L}im_u(\widetilde{Z}_{1,g})\mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \sqrt{p}\widetilde{Z}_{2,g})$ appearing in (15) can be seen as the heat kernel at the time u of some explicit harmonic oscillator (depending on $\widetilde{Z}_{1,g}$) on $\mathbb{R}^{2n}$, evaluated at $(\sqrt{p}g^{-1}\widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g})$. For more details see (49) and (52).

Remark 3 From Theorem 4, for $x \in M_{sing}$, we have $\left|p^{-n}e^{-\frac{u}{p}\square_p}(x,x) - |G_x|\mathcal{L}im_u(x)\right| \leq Cp^{-1/2}$. In particular, unlike in the usual non-singular case, if M_{sing} is not empty, the asymptotics of Theorem 3 cannot be uniform on M_{reg}.

An analogous result holds true for the Bergman kernel, see [20] or [9].

Next, we give an application of the inequalities (6). The class of Moishezon orbifolds is defined in a similar way as usual regular Moishezon manifolds: we begin by defining meromorphic functions on an orbifold as local quotient of holomorphic orbifold functions, and we say that a compact connected orbifold M is a *Moishezon orbifold* if it possesses $\dim M$ meromorphic functions that are algebraically independent, i.e., they satisfy no non-trivial polynomial equation (see Sect. 5.1 for details).

For regular Moishezon manifold, Grauert and Riemenschneider [14] conjectured that if a compact connected manifold M possesses a smooth Hermitian bundle which is semi-positive everywhere and positive on an open dense set, then M is Moishezon. Siu proved a stronger version of this conjecture in [23,24], and Demailly improve Siu's result in [10]. Here, we prove that these results are still valid for orbifolds.

Theorem 5 *Let M be a compact connected complex orbifold of dimension n and let (L, h^L) be a holomorphic orbifold line bundle on M. If one of the following conditions holds:*

(i) *(Siu-type criterion) (L, h^L) is semi-positive and positive at one point,*
(ii) *(Demailly-type criterion) (L, h^L) satisfies*

$$\int_{M(\leq 1)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n > 0, \tag{16}$$

then M is a Moishezon orbifold.

Note that in the course of the proof of Theorem 5, we prove an orbifold version of the famous Siegel's Lemma [22] (see also [20, Lemma 2.2.6]), see Theorem 9.

This paper is organized as follows. In Sect. 2 we recall the definitions and basic properties of orbifolds. In Sect. 3 we study the convergence of the heat kernel and prove Theorems 3 and 4. In Sect. 4 we use these asymptotic result to prove the holomorphic Morse inequalities (Theorem 1). Finally, in Sect. 5, we use these inequalities to give a geometric criterion for a compact connected orbifold to be a Moishezon orbifold (Theorem 5).

2 Background on orbifolds

In this section we recall the background about orbifold. The content of this section is essentially taken from [20] and [21].

2.1 Definitions

We first define a category $\mathcal{M}_s$ as follows.

Objects: classes of pairs (G, M) where M is a connected smooth manifold and G is a finite group acting effectively on M (i.e., the unit is the unique element of G acting as Id_M);

Morphisms: a morphism $\Phi\colon (G, M) \to (G', M')$ is a family of open embeddings $\{\varphi\colon M \to M'\}_{\varphi\in\Phi}$ satisfying:

1. for each $\varphi \in \Phi$ there is an injective group morphism $\lambda_\varphi : G \hookrightarrow G'$ for which φ is equivariant, i.e., $\varphi(g.x) = \lambda_\varphi(g).\varphi(x)$ for $x \in M$ and $g \in G$;
2. for $g' \in G'$ and $\varphi \in \Phi$, we have $g'.(\varphi(M)) \cap \varphi(M) \neq \emptyset \Longrightarrow g' \in \lambda_\varphi(G)$;
3. for any $\varphi \in \Phi$, we have $\Phi = \{g'\varphi,\ g' \in G'\}$, where $g'\varphi : x \in M \mapsto g'.\varphi(x) \in M'$.

Definition 1 (Orbifold chart, atlas, structure) Let M be a paracompact Hausdorff space.

An m-dimensional *orbifold chart* on M consists of a connected open set U of M, an object $(G_U, \widetilde{U})$ of $\mathcal{M}_s$ with $\dim \widetilde{U} = m$, and a ramified covering $\tau_U : \widetilde{U} \to U$ which is G_U-invariant and induces a homeomorphism $U \simeq \widetilde{U}/G_U$. We denote the chart by $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$.

An m-dimensional *orbifold atlas* $\mathcal{V}$ on M consists of a family of m-dimensional orbifold charts $\mathcal{V}(U) = ((G_U, \widetilde{U}) \xrightarrow{\tau_U} U)$ satisfying the following conditions:

1. the open sets $U \subset M$ form a covering $\mathcal{U}$ such that:
$$\text{for any } U, U' \in \mathcal{U} \text{ and } x \in U \cap U', \text{ there exists } U'' \in \mathcal{U} \text{ such that } x \in U'' \subset U \cap U'. \tag{17}$$
2. for any $U, V \in \mathcal{U}$ with $U \subset V$ there exists a morphism (of $\mathcal{M}_s$) $\Phi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$ which covers the inclusion $U \subset V$ and satisfies $\Phi_{WU} = \Phi_{WV} \circ \Phi_{VU}$ for any $U, V, W \in \mathcal{U}$ with $U \subset V \subset W$. These morphisms are called *restriction morphisms*.

It is easy to see that there exists a unique maximal orbifold atlas $\mathcal{V}_{max}$ containing $\mathcal{V}$: it consists of all orbifold charts $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$ which are locally isomorphic to charts from $\mathcal{V}$ in the neighborhood of each point of M. A maximal orbifold atlas $\mathcal{V}_{max}$ is called an *orbifold structure* and the pair $(M, \mathcal{V}_{max})$ is called an *orbifold*. As usual, once we have an orbifold atlas $\mathcal{V}$ on M, it uniquely determines a maximal atlas $\mathcal{V}_{max}$ and we denote the corresponding orbifold simply by $(M, \mathcal{V})$.

In the above definition, we can replace $\mathcal{M}_s$ with a category of manifolds with an additional structure such as orientation, Riemannian metric, almost-complex structure or complex structure. In this case we require that the morphisms and the groups preserve the specified structure. In this way we can define oriented, Riemannian, almost-complex or complex orbifolds.

Certainly, for any object $(G_U, \widetilde{U})$ of $\mathcal{M}_s$, we can always construct a G-invariant Riemannian metric on $\widetilde{U}$. By a partition of unity argument, there always exists a Riemannian metric on an given orbifold $(M, \mathcal{V})$.

Remark 4 Let P be a smooth manifold, and let H be a compact Lie group acting locally freely on P. Then the quotient space P/H is an orbifold. Reciprocally, any orbifold M can be presented by this way, with $H = O(n)$ ($n = \dim M$) see [17, p. 76] and [18, p. 144].

Definition 2 (*Regular and singular set*) Let $(M, \mathcal{V})$ be an orbifold. For each $x \in M$, we can choose a small neighborhood $(G_x, \widetilde{U}_x) \to U_x$ such that $x \in \widetilde{U}_x$ is a fixed point of G_x (such G_x is unique up to isomorphisms for each $x \in M$ from the definition). If the cardinal $|G_x|$ of G_x is 1, then x is a regular point of M, meaning that M is a smooth manifold in a neighborhood of x. If $|G_x| > 1$, then x is a singular point of M. We denote by $M_{sing} = \{x \in M : |G_x| > 1\}$ the singular set of M and by $M_{reg} = M \backslash M_{sing}$ the regular set of M.

The following lemma is proved in [20, Lemma 5.4.3].

Lemma 1 *With the above notations, we can choose the local coordinates $\widetilde{U}_x \subset \mathbb{R}^m$ such that the finite group G_x acts linearly on $\mathbb{R}^m$ and $\{0\} = \tau_x^{-1}(x)$.*

In the sequel we will always use such charts.

Definition 3 (*Orbifold vector bundle*) An orbifold vector bundle E over an orbifold $(M, \mathcal{V})$ is defined as follows: E is an orbifold and for $U \in \mathcal{U}$, $(G_U^E, \widetilde{p}_U : \widetilde{E}_U \to \widetilde{U})$ is a G_U^E-equivariant vector bundle where $(G_U^E, \widetilde{E}_U)$ gives the orbifold structure of E and $(G_U = G_U^E/K_U^E, \widetilde{U})$, $K_U^E = \ker\big(G_U^E \to \mathrm{Diffeo}(\widetilde{U})\big)$, gives the orbifold structure on M. We also require that for any $V \subset U$, each embedding in the restriction morphism of E is an isomorphism of equivariant vector bundles compatible with an embedding in the restriction morphism of X.

If moreover G_U^E acts effectively on $\widetilde{U}$ for $U \in \mathcal{U}$, i.e. $K_U^E = \{1\}$, we call E a proper orbifold vector bundle.

Let E be an orbifold vector bundle on $(M, \mathcal{V})$. For $U \in \mathcal{U}$, let $\widetilde{E_U^{pr}}$ be the maximal K_U^E-invariant sub-bundle of $\widetilde{E}_U$ on U. Then $(G_U, \widetilde{E_U^{pr}})$ defines a proper orbifold vector bundle on $(M, \mathcal{V})$, which is denoted by E^{pr}.

Example 1 The (proper) orbifold tangent bundle TM of an orbifold M is defined by $(G_U, T\widetilde{U} \to \widetilde{U})$, for $U \in \mathcal{U}$.

Definition 4 ($\mathscr{C}^k$ **section**) Let $E \to M$ be an orbifold bundle. A section $s\colon M \to E$ is called $\mathscr{C}^k$, for $k \in \mathbb{N} \cup \{\infty\}$, if for each $U \in \mathcal{U}$, $s_{|U}$ is covered by a G_U^E-invariant $\mathscr{C}^k$ section $\widetilde{s}_U : \widetilde{U} \to \widetilde{E}_U$. We denote by $\mathscr{C}^k(M, E)$ the space of $\mathscr{C}^k$ sections of E on M.

When it entails no confusion, we will denote $\widetilde{s}_U$ simply by s.

Remark 5 A smooth object on M as a section, a Riemannian metric, a complex structure, etc... can be seen as an usual regular object on M_{reg} such that its lift in any chart of the orbifold atlas can be extended to a smooth corresponding object.

Definition 5 (*Integration*) If M is oriented, we define the integral $\int_M \omega$ for a form ω over M (i.e. a section of $\Lambda^\bullet(T^*M)$ over M) as follows: if $\mathrm{supp}(\omega) \subset U \in \mathcal{U}$, then

$$\int_M \omega = \frac{1}{|G_U|}\int_{\widetilde{U}} \widetilde{\omega}_U. \tag{18}$$

It is easy to see that the definition is independent of the chart. For general ω we extend the definition by using a partition of unity.

Note also that if M is a Riemannian orbifold, there exists a canonical volume element dv_M on M, which is a section of $\Lambda^{\dim M}(T^*M)$. Hence, we can also integrate functions on M.

Definition 6 (*Metric structure on Riemannian orbifold*) Let $(M, \mathcal{V})$ be a compact Riemannian orbifold. For $x, y \in M$, we define $d^M(x, y)$ by:

$$\begin{aligned} d^M(x, y) = \inf_\gamma \Big\{ \textstyle\sum_i \int_{t_{i-1}}^{t_i} |\frac{\partial}{\partial t}\tilde{\gamma}_i(t)|dt \;\Big|\; & \gamma\colon [0,1] \to M,\ \gamma(0) = x,\ \gamma(1) = y,\ \text{such that} \\ \text{there are } t_0 = 0 < t_1 < \cdots < t_k = 1 \text{ with } & \gamma([t_{i-1}, t_i]) \subset U_i,\ U_i \in \mathcal{U}, \\ \text{and a } \mathscr{C}^\infty \text{ map } \tilde{\gamma}_i : [t_{i-1}, t_i] \to \widetilde{U}_i & \text{ which covers } \gamma|_{[t_{i-1},t_i]} \Big\}. \end{aligned} \tag{19}$$

Then (M, d) is a metric space.

2.2 Kernels on orbifolds

Let $(M, \mathcal{V})$ be a Riemannian orbifold and let E be a proper orbifold vector bundle on M.

For any orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, $U \in \mathcal{U}$, we will add a tilda $\tilde{}$ to objects on U to indicate the corresponding objects on $\widetilde{U}$.

Consider a section $\widetilde{\mathcal{K}} \in \mathscr{C}^\infty(\widetilde{U} \times \widetilde{U}, \mathrm{pr}_1^* \widetilde{E} \otimes \mathrm{pr}_2^* \widetilde{E}^*)$ such that

$$(g, 1)\widetilde{\mathcal{K}}(g^{-1}\tilde{x}, \tilde{x}') = (1, g^{-1})\widetilde{\mathcal{K}}(\tilde{x}, g\tilde{x}') \quad \text{for any } g \in G_U, \tag{20}$$

where the action of $G_U \times G_U$ on $\widetilde{E}_{\tilde{x}} \otimes \widetilde{E}^*_{\tilde{x}'}$ is given by $(g_1, g_2).u \otimes \xi = (g_1 u) \otimes (g_2 \xi)$. We can then define an operator $\widetilde{\mathcal{K}} \colon \mathscr{C}_0^\infty(\widetilde{U}, \widetilde{E}) \to \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ by

$$(\widetilde{\mathcal{K}}\tilde{s})(\tilde{x}) = \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\tilde{x}, \tilde{x}')\tilde{s}(\tilde{x}')dv_{\widetilde{U}}(\tilde{x}') \quad \text{for } \tilde{s} \in \mathscr{C}_0^\infty(\widetilde{U}, \widetilde{E}). \tag{21}$$

Recall that a section $s \in \mathscr{C}^\infty(U, E)$ is identified with a G_U-invariant section $\tilde{s} \in \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$. Thus, we can define an operator $\mathcal{K} \colon \mathscr{C}_0^\infty(U, E) \to \mathscr{C}^\infty(U, E)$ by

$$(\mathcal{K}s)(x) = \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\tilde{x}, \tilde{x}')\tilde{s}(\tilde{x}')dv_{\widetilde{U}}(\tilde{x}') \quad \text{for } s \in \mathscr{C}_0^\infty(U, E), \tag{22}$$

where $\tilde{x} \in \tau_U^{-1}(x)$. Then the smooth kernel $\mathcal{K}(x, x')$ of the operator $\mathcal{K}$ with respect to dv_M is given by

$$\mathcal{K}(x, x') = \sum_{g \in G_U} (g, 1)\widetilde{\mathcal{K}}(g^{-1}\tilde{x}, \tilde{x}'). \tag{23}$$

2.3 Complex orbifolds and Dolbeault cohomology

Let M be a compact complex orbifold of complex dimension n and with complex structure J. Let E be a holomorphic orbifold vector bundle on M.

Let $\mathscr{O}_M$ be the sheaf over M of local G_U-invariant holomorphic functions over $\widetilde{U}$, for $U \in \mathcal{U}$. An element of $\mathscr{O}_M(M)$ is called an *orbifold holomorphic function* on M.

Likewise, the local G_U^E-invariant sections of $\widetilde{E}$ over $\widetilde{U}$ define a sheaf $\mathscr{O}_M(E)$ over M. Let $H^\bullet(M, \mathscr{O}_M(E))$ be the cohomology of this sheaf. Notice that, by the definition, we have $\mathscr{O}_M(E) = \mathscr{O}_M(E^{pr})$. Thus without lost generality, we may and will assume that E is a proper orbifold vector bundle on M.

Consider a section $s \in \mathscr{C}^\infty(M, E)$ and a local section $\tilde{s} \in \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ covering s over U. Then $\bar{\partial}^{\widetilde{E}}\tilde{s}$ covers a section of $T^{*(0,1)}X \otimes E$ over U, denoted by $\bar{\partial}^E s|_U$. The sections $\bar{\partial}^E s|_U$ for $U \in \mathcal{U}$ patch together to define a global section $\bar{\partial}^E s$ of $T^{*(0,1)}X \otimes E$ over M. In a similar way, we can define $\bar{\partial}^E \alpha$ for $\alpha \in \Omega^{\bullet,\bullet}(M, E) := \mathscr{C}^\infty(M, \Lambda^{\bullet,\bullet}(T^*M) \otimes E)$. We thus obtain the Dolbeault complex

$$0 \to \Omega^{0,0}(M, E) \xrightarrow{\bar{\partial}^E} \cdots \xrightarrow{\bar{\partial}^E} \Omega^{0,n}(M, E) \to 0. \tag{24}$$

From the abstract de Rham theorem, there exists a canonical isomorphism (for more details, see [20, Section 5.4.2])

$$H^\bullet(\Omega^{0,\bullet}(M, E), \bar{\partial}^E) \simeq H^\bullet(M, \mathscr{O}_M(E)). \tag{25}$$

In the sequel, we will denote both these cohomology groups simply by $H^\bullet(M, E)$.

Let g^{TM} be a Riemannian metric on TM, with associated volume form dv_M, and let h^E be a Hermitian metric on E. They induce a L^2 Hermitian product $\langle \cdot, \cdot \rangle$ on $\Omega^\bullet(M, E)$ given by

$$\langle s_1, s_2\rangle = \int_M \langle s_1(x), s_2(x)\rangle_{\Lambda^{0,\bullet}(T^*X)\otimes E}\, dv_M(x). \tag{26}$$

Let $\bar{\partial}^{E,*}$ be the formal adjoint of $\bar{\partial}^E$ for this L^2 product, and let

$$\begin{aligned} D^E &= \sqrt{2}(\bar{\partial}^E + \bar{\partial}^{E,*}),\\ \square^E &= \frac{1}{2}D^{E,2} = \bar{\partial}^E\bar{\partial}^{E,*} + \bar{\partial}^{E,*}\bar{\partial}^E. \end{aligned} \tag{27}$$

be, respectively, the associated Dolbeault-Dirac operator and Kodaira Laplacian. Then $\square^E$ a differential operator of order 2 acting on sections of $\Lambda^\bullet(T^*M)\otimes E$ (i.e., on each U, it is covered by a $G_U^{\Lambda^\bullet(T^*M)\otimes E}$-invariant differential operator of order 2 acting on $\mathscr{C}^\infty(\widetilde{U}, \Lambda^\bullet(\widetilde{T^*M})\otimes\widetilde{E})$), which is formally self-adjoint and elliptic. Moreover, it preserves the $\mathbb{Z}$-grading on $\Omega^\bullet(M,E)$.

A crucial point is that classical Hodge theory still holds in the present orbifold setting, see [19, Proposition 2.2].

Theorem 6 (Hodge theory) *For any $q\in\mathbb{N}$, we have the following orthogonal decomposition*

$$\Omega^{0,q}(M,E) = \mathrm{Ker}(D^E|_{\Omega^{0,q}}) \oplus \mathrm{Im}(\bar{\partial}^E|_{\Omega^{0,q-1}}) \oplus \mathrm{Im}(\bar{\partial}^{E,*}|_{\Omega^{0,q+1}}). \tag{28}$$

In particular, for $q\in\mathbb{N}$, we have the canonical isomorphism

$$\mathrm{Ker}(D^E|_{\Omega^{0,q}}) = \mathrm{Ker}(\square^E|_{\Omega^{0,q}}) \simeq H^q(M,E). \tag{29}$$

Note that on orbifolds, we still have a unique Hermitian and holomorphic connection ∇^E associated with E and h^E. We call it the Chern connection of (E, h^E).

3 Convergence of the heat kernel

In the sequel, when we define objects on orbifolds, one can always think of them as being defined by standard objects which are G_U-invariant on each chart $(G_U, \widetilde{U})$.

Let M be a compact complex orbifold of dimension n. We denote the complex structure of M by J. Let (L, h^L) an orbifold Hermitian holomorphic line bundle on M and let (E, h^E) be an orbifold Hermitian holomorphic vector bundle on M. Recall that by Sect. 2.3 we may assume that L and E are proper. We denote the corresponding Chern connections by ∇^L and ∇^E respectively, and we denote their curvatures by R^L and R^E.

We define the orbifold bundles $\mathbb{E}$ and $\mathbb{E}_p$ over M by

$$\begin{aligned} \mathbb{E} &= \Lambda^{0,\bullet}(T^*M)\otimes E,\\ \mathbb{E}_p &= \Lambda^{0,\bullet}(T^*M)\otimes E\otimes L^p. \end{aligned} \tag{30}$$

Let g^{TM} be a Riemannian metric on TM which is compatible with J. Then $\mathbb{E}$ and $\mathbb{E}_p$ are naturally equipped with the Hermitian metrics $h^{\mathbb{E}}$ and $h^{\mathbb{E}_p}$ induced by g^{TM}, h^L and h^E. We endow $\mathscr{C}^\infty(M,\mathbb{E}_p)$ with the L^2 scalar product associated with g^{TM}, h^L and h^E as in (26). As in Sect. 2.3, we can define $D^{L^p\otimes E}$ and $\square^{L^p\otimes E}$. We will denote these operators respectively by D_p and $\square_p$ for short.

Let $e^{-u\square_p}$ be the heat kernel of $\square_p$ and let $e^{-u\square_p}(x,x')$ be its smooth kernel with respect to $dv_M(x')$. Concerning heat kernels on orbifolds, we refer the reader to [19, Section 2.1].

In this section, we study the convergence as $p\to+\infty$ of the heat kernel. We follow the approach of [20].

3.1 Localisation

Let $\varepsilon > 0$ be a small number (smaller than the quarter of the injectivity radius of (M, g^{TM})). Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$f(t) = \begin{cases} 1 & \text{for } |t| < \varepsilon/2, \\ 0 & \text{for } |t| > \varepsilon. \end{cases} \tag{31}$$

For $u > 0$ and $a \in \mathbb{C}$, set

$$\begin{aligned} \mathrm{F}_u(a) &= \int_{\mathbb{R}} e^{iva} \exp(-v^2/2) f(v\sqrt{u}) \frac{dv}{\sqrt{2\pi}}, \\ \mathrm{G}_u(a) &= \int_{\mathbb{R}} e^{iva} \exp(-v^2/2)(1 - f(v\sqrt{u})) \frac{dv}{\sqrt{2\pi}}. \end{aligned} \tag{32}$$

These functions are even holomorphic functions. Moreover, the restrictions of F_u and G_u to $\mathbb{R}$ lie in the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$\mathrm{F}_u(vD_p) + \mathrm{G}_u(vD_p) = \exp\left(-\frac{v^2}{2} D_p^2\right) \quad \text{for } v > 0. \tag{33}$$

Let $\mathrm{G}_u(vD_p)(x, x')$ be the smooth kernel of $\mathrm{G}_u(vD_p)$ with respect to $dv_M(x')$. Then for any $m \in \mathbb{N}$, $u_0 > 0$, $\varepsilon > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $u > u_0$ and any $p \in \mathbb{N}^*$,

$$\left| \mathrm{G}_{\frac{u}{p}} \left(\sqrt{u/p} D_p \right) (\cdot, \cdot) \right|_{\mathscr{C}^m(M \times M)} \le C p^N \exp\left(-\frac{\varepsilon^2 p}{16u} \right). \tag{34}$$

This is proved in the same way as [20, Proposition 1.6.4], the only difference is that when we use Sobolev norms or inequality on an open U, we in fact have to use them on $\widetilde{U}$ for the pulled-back operators.

As pointed out in [19, Section 6.6], the property of finite propagation speed of solutions of hyperbolic equations still holds on an orbifold (see the proof in [20, Appendix D.2]). Thus, $\mathrm{F}_{\frac{u}{p}} \left(\sqrt{u/p} D_p \right)(x, x')$ vanishes if $d^M(x, x') \ge \varepsilon$ and $\mathrm{F}_{\frac{u}{p}} \left(\sqrt{u/p} D_p \right)(x, \cdot)$ only depends on the restriction of D_p to the ball $B^M(x, \varepsilon)$. This, together with (33) and (34), implies that the problem of the asymptotic of $e^{-u\square_p}(x, \cdot)$ is local.

More precisely, this means that, for $x_0 \in M$ fixed, one can trivialize the various bundles over U_{x_0} and replace M by $M_0 = \mathbb{C}^n / G_{x_0} \supset \widetilde{U}_{x_0} / G_{x_0} = U_{x_0}$ (using a local chart as in Lemma 1). Then we construct a metric g^{TM_0} on M_0 and an operator L_{p,x_0} acting on $\mathbb{E}_{p,x_0}$ over M_0 such that g^{TM_0} (resp. L_{p,x_0}) coincides with g^{TM} (resp. $\square_p$) near $x_0 = 0$ and such that its lift $\widetilde{g^{TM_0}}$ (resp. $\widetilde{L}_{p,x_0}$) on $\widetilde{M}_0 = \mathbb{C}^n$ is the constant metric $(\widetilde{g^{TM}{}_{U_{x_0}}})_{\tilde{x}=0}$ (resp. the usual Laplacian on $\mathbb{C}^n$ for this metric) away from 0. Then we can approximate the heat kernel of $\square_p$ by the one of L_{p,x_0}, see equation (42) below.

We now give the details of these constructions, following [20, Section 1.6.3].

By [20, (1.2.61) and (1.4.27)], the Levi–Civita connection ∇^{TM} on (M, g^{TM}), the complex structure J of M and the Chern connection ∇^E induce a Hermitian connexion $\nabla^{B,\mathbb{E}}$ on $(\mathbb{E}, h^{\mathbb{E}})$ which preserve the $\mathbb{Z}$-grading. This connection is called the Bismut connection. Let $\Delta^{B,\mathbb{E}}$ be the associated Bochner Laplacian, that is

$$\Delta^{B,\mathbb{E}} = -\sum_{i=1}^{2n} \left((\nabla_{e_i}^{B,\mathbb{E}})^2 - \nabla_{\nabla_{e_i}^{TM} e_i}^{B,\mathbb{E}} \right), \tag{35}$$

with $\{e_i\}$ an orthonormal frame of TM.

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$ (with the metric induced by g^{TM}) with dual frame $\{w^j\}_{j=1}^n$. Set

$$\begin{aligned} \omega_d &= -\sum_{k,\ell} R^L(w_k, \overline{w}_\ell)\overline{w}^\ell \wedge i_{\overline{w}_k}, \\ \tau &= \sum_j R^L(w_j, \overline{w}_j). \end{aligned} \tag{36}$$

From Bismut's Lichnerowicz formula (see [20, Theorem 1.4.7]), which still holds here because the computations involved are local and hence can be carried on orbifold charts, there exists a self-adjoint section Φ_E of $\mathrm{End}(\Lambda^{0,\bullet}(T^*M) \otimes E))$ such that

$$\Box_p = \frac{1}{2}\Delta^{B,\mathbb{E}} + \omega_d + \frac{1}{2}\tau + \Phi_E. \tag{37}$$

We fix $x_0 \in M$ and $\varepsilon > 0$ smaller than the quarter of the injectivity radius of M. In the sequel we will use a local chart $(G_{x_0}, \widetilde{U}_{x_0})$ near x_0 as in Lemma 1. Note that we then have $T_{x_0}M \simeq \mathbb{C}^n/G_{x_0}$.

We denote by $B^M(x_0, 4\varepsilon)$ and $B^{T_{x_0}M}(0, 4\varepsilon)$ the open balls in M and $T_{x_0}M$ with center x_0 and 0 and radius 4ε, respectively. The exponential map $T_{x_0}M \ni Z \mapsto \exp_{x_0}^M(Z) \in M$ is a diffeomorphism from $B^{T_{x_0}M}(0, 4\varepsilon)$ on $B^M(x_0, 4\varepsilon)$. From now on, we identify $B^{T_{x_0}M}(0, \varepsilon)$ and $B^M(x_0, 4\varepsilon)$.

For $Z \in B^{T_{x_0}M}(0, 4\varepsilon)$, we identify (L_Z, h_Z^L) and $(\mathbb{E}_Z, h_Z^{\mathbb{E}})$ to $(L_{x_0}, h_{x_0}^L)$ and $(\mathbb{E}_{x_0}, h_{x_0}^{\mathbb{E}})$ by the parallel transport with respect to ∇^L and $\nabla^{B,\mathbb{E}}$ along the ray $u \in [0,1] \mapsto uZ$. We denote the corresponding connection forms by Γ^L and $\Gamma^{\mathbb{E}}$. Note that Γ^L and $\Gamma^{\mathbb{E}}$ are skew-adjoint with respect to $h_{x_0}^L$ and $h_{x_0}^{\mathbb{E}}$.

Let $\rho\colon \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = 1 \text{ if } |v| < 2 \quad \text{and} \quad \rho(v) = 0 \text{ if } |v| > 4. \tag{38}$$

We denote by $\nabla_{\widetilde{V}}$ the ordinary differentiation operator on $\widetilde{M}_0 = \mathbb{C}^n$ in the direction $\widetilde{V}$. Then ∇ is G_{x_0}-equivariant since G_{x_0} acts linearly on $\mathbb{C}^n$, thus it induces an operator still denoted by ∇ on $M_0 = \mathbb{C}^n/G_{x_0}$. Set

$$\nabla^{\mathbb{E}_{p,x_0}} = \nabla + \rho(|Z|/\varepsilon)(p\Gamma^L + \Gamma^{\mathbb{E}})(Z), \tag{39}$$

which is a Hermitian connection on the trivial bundle $(\mathbb{E}_{p,x_0}, h_{x_0}^{\mathbb{E}_p})$ over $M_0 \simeq T_{x_0}M$, where the identification is given by

$$[Z_1, \ldots, Z_{2n}] \in \mathbb{R}^{2n}/G_{x_0} \mapsto \left[\sum_{i=1}^{2n} Z_i \widetilde{e}_i\right] \in T_{x_0}M. \tag{40}$$

Here, $\{\widetilde{e}_i\}_i$ is an orthonormal basis of $(\widetilde{TM_{U_{x_0}}})_{\tilde{x}=0}$.

Let g^{TM_0} be the metric on M_0 which coincides with g^{TM} on $B^{T_{x_0}M}(0, 2\varepsilon)$ and such that its lift $\widetilde{g^{TM_0}}$ on $\widetilde{M}_0$ is the constant metric $(\widetilde{g^{TM}}_{U_{x_0}})_{\tilde{x}=0}$ outside of $B^{\widetilde{M}_0}(0, 4\varepsilon)$. Let $dv_{\widetilde{M}_0}$ be the Riemannian volume of $\widetilde{g^{TM_0}}$.

Let $\Delta^{\mathbb{E}_{p,x_0}}$ be the Bochner Laplacian associated with $\nabla^{\mathbb{E}_{p,x_0}}$ and g^{TM_0}. Set

$$L_{p,x_0} = \frac{1}{2}\Delta^{\mathbb{E}_{p,x_0}} - p\rho(|Z|/\varepsilon)\left(\omega_{d,Z} + \frac{1}{2}\tau_Z\right) - \rho(|Z|/\varepsilon)\Phi_{E,Z}. \tag{41}$$

Then L_{p,x_0} is a self-adjoint operator for the L^2-product induced by $h^{\mathbb{E}_p}_{x_0}$ and g^{TM_0}, and coincides with $\square_p$ on $B^{T_{x_0}M}(0, 2\varepsilon)$. We denote by $\widetilde{L}_{p,x_0}$ the G_{x_0}-invariant lift of L_{p,x_0} on $\widetilde{M}_0$.

From the above discussion, and using the same technics as in [20, Lemma 1.6.5] (because, as said before, the property of finite propagation speed of solutions of hyperbolic equations still holds on orbifolds) we can then prove that for $(x, x') \in B^M(x_0, \varepsilon/2)$ corresponding to $(Z, Z') \in M_0$, there exist $C > 0$ and $K \in \mathbb{N}$ such that

$$\left| e^{-\frac{u}{p}\square_p}(x, x') - e^{-\frac{u}{p}L_{p,x_0}}(Z, Z') \right| \leq Cp^K e^{-\frac{\varepsilon^2 p}{16u}}, \tag{42}$$

where $e^{-\frac{u}{p}L_{p,x_0}}(Z, Z')$ is the smooth kernel of $e^{-\frac{u}{p}L_{p,x_0}}$ with respect to $dv_{M_0}(Z')$, the Riemannian volume of g^{TM_0}.

Now, as in Sect. 2.2, we have

$$e^{-\frac{u}{p}L_{p,x_0}}(Z, Z') = \sum_{g \in G_{x_0}} (g, 1) e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(g^{-1}\widetilde{Z}, \widetilde{Z}'), \tag{43}$$

where $e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(\tilde{Z}, \tilde{Z}')$ is the smooth kernel of $e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}$ with respect to $dv_{\widetilde{M}_0}(\tilde{Z}')$.

Thus, from (42) and (43), we have to study the asymptotic of $e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(\tilde{Z}, \tilde{Z}')$, which is a kernel on a honest vector space. Note that, even if we want to restrict $e^{-\frac{u}{p}\square_p}$ to the diagonal, we have to study the off-diagonal asymptotic of $e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}$.

3.2 Rescaling

As in [20], we will rescale the variables in $\widetilde{M}_0$.

In this paragraph, we work on $\widetilde{U}_{x_0}$ and for any bundle F on M we will denote $\widetilde{F}_{U_{x_0}}$ simply by $\widetilde{F}$. Likewise, we will drop the subscript U_{x_0} in the notations of lifts to $\widetilde{F}$ of objects on F.

Let $S_{\widetilde{L}}$ be a G_{x_0}-invariant unit vector of $\widetilde{L}|_0$. Using $S_{\widetilde{L}}$ and the above discussion, we get an isometry $\widetilde{\mathbb{E}}_{p,x_0} \simeq \widetilde{\mathbb{E}}_{x_0}$. Thus, $\widetilde{L}_{p,x_0}$ can be seen as an operator on $\widetilde{\mathbb{E}}_{x_0}$. Note that our formulas will not depend on the choice of S_L as the isomorphism $\mathrm{End}(\widetilde{\mathbb{E}}_{p,x_0}) \simeq \mathrm{End}(\widetilde{\mathbb{E}}_{x_0})$ is canonical.

Let $dv_{\widetilde{TM}}$ be the Riemannian volume of $(\widetilde{M}_0, \widetilde{g^{TM}}|_0)$. Let $\widetilde{\kappa}$ be the smooth positive function defined by

$$dv_{\widetilde{M}_0}(\widetilde{Z}) = \widetilde{\kappa}(\widetilde{Z}) dv_{\widetilde{TM}}(\widetilde{Z}), \tag{44}$$

with $\widetilde{\kappa}(0) = 1$. Note that this definition is compatible with (13) near 0, which will be *in fine* the only region of interest.

Let $R^{\widetilde{L}}$ be the Chern curvature of $(\widetilde{L}, \widetilde{h^L})$. Let $\widetilde{\omega}_d$ and $\widetilde{\tau}$ be defined from $R^{\widetilde{L}}$ as ω_d and τ were defined from R^L in (36). Then $\widetilde{\omega}_d$ and $\widetilde{\tau}$ are in fact the lifts of ω_d and τ.

Recall that ∇_V is the ordinary differentiation operator on $\widetilde{M}_0 = \mathbb{C}^n$ in the direction V.

We will now make the change of parameter $t = \frac{1}{\sqrt{p}} \in]0, 1]$.

Definition 7 For $s \in \mathscr{C}^\infty(\widetilde{M}_0, \mathbb{E}_{x_0})$ and $\tilde{Z} \in \mathbb{C}^n$ set

$$
\begin{aligned}
(S_t s)(\tilde{Z}) &= s(\tilde{Z}/t),\\
\nabla_0 &= \nabla + \frac{1}{2} R_0^{\widetilde{L}}(\tilde{Z}, \cdot),\\
\widetilde{\mathscr{L}_t} &= t^2 S_t^{-1} \widetilde{\kappa}^{1/2} \widetilde{L}_{p,x_0} \widetilde{\kappa}^{-1/2} S_t,\\
\widetilde{\mathscr{L}_0} &= -\frac{1}{2} \sum_{i=1}^{2n} \left(\nabla_{0,\tilde{e}_i}\right)^2 - \widetilde{\omega}_{d,0} - \frac{1}{2} \widetilde{\tau}_0.
\end{aligned}
\tag{45}
$$

Then, exactly as in [20, Lemma 1.6.6], our constructions imply that $\widetilde{\mathscr{L}_t} = \widetilde{\mathscr{L}_0} + O(t)$.

Let $e^{-u\widetilde{\mathscr{L}_t}}(\widetilde{Z}, \widetilde{Z}')$ be the smooth kernel of $e^{-u\widetilde{\mathscr{L}_t}}$ with respect to $dv_{\widetilde{TM}}(\widetilde{Z}')$ (for $t > 0$ or $t = 0$). Now, as we are working on a vector space, we can apply all the results of [9] (see also [20, Section 4.2.2]) to $\widetilde{\mathscr{L}_t}$ and $\widetilde{\mathscr{L}_0}$, and we get the following full off-diagonal convergence (see [9] or [20, Theorem 4.2.8]).

Theorem 7 *There exist $C, C' > 0$ and $N \in \mathbb{N}$ such that for any $m, m' \in \mathbb{N}$ and $u_0 > 0$, there are $C_{m,m'} > 0$ and $N \in \mathbb{N}$ such that for any $t \in]0, t_0]$, $u \geq u_0$ and $\widetilde{Z}, \widetilde{Z}' \in \widetilde{M}_0$ with $|\widetilde{Z}|, |\widetilde{Z}'| \leq 1$*

$$
\begin{aligned}
&\sup_{|\alpha|,|\alpha'|\leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial \widetilde{Z}^\alpha \partial \widetilde{Z}'^{\alpha'}} \left(e^{-u\widetilde{\mathscr{L}_t}} - e^{-u\widetilde{\mathscr{L}_0}} \right)(\widetilde{Z}, \widetilde{Z}') \right|_{\mathscr{C}^{m'}(M)}\\
&\quad \leq C_{m,m'} t \left(1 + |\widetilde{Z}| + |\widetilde{Z}'|\right)^N \exp\left(Cu - \frac{C'}{u} |\widetilde{Z} - \widetilde{Z}'|^2 \right),
\end{aligned}
\tag{46}
$$

where $|\cdot|_{\mathscr{C}^{m'}(M)}$ denotes the $\mathscr{C}^{m'}$-norm with respect to the parameter $x_0 \in M$ used to define the operators $\widetilde{\mathscr{L}_t}$ and $\widetilde{\mathscr{L}_0}$ on $\mathbb{C}^n$.

3.3 Conclusion

From (45), a change of variable gives that

$$
e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(\widetilde{Z}, \widetilde{Z}') = p^n e^{-u\widetilde{\mathscr{L}_t}}(\widetilde{Z}/t, \widetilde{Z}'/t) \widetilde{\kappa}^{-1/2}(\widetilde{Z}) \widetilde{\kappa}^{-1/2}(\widetilde{Z}'). \tag{47}
$$

Thus, with Theorem 7, we infer that for any multi-index α with $|\alpha| \leq m$ and for $|\widetilde{Z}|$ small,

$$
\begin{aligned}
&\left| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha} \left(p^{-n} e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(g^{-1}\widetilde{Z}, \widetilde{Z}) - e^{-u\widetilde{\mathscr{L}_0}}(\sqrt{p} g^{-1}\widetilde{Z}, \sqrt{p}\widetilde{Z}) \widetilde{\kappa}^{-1}(\widetilde{Z}) \right) \right|_{\mathscr{C}^{m'}(M)}\\
&\quad \leq C p^{\frac{m-1}{2}} \left(1 + \sqrt{p}|\widetilde{Z}|\right)^N e^{-cp|\widetilde{Z} - g^{-1}\widetilde{Z}|^2}.
\end{aligned}
\tag{48}
$$

We define $\dot{R}^{\widetilde{L}} \in \mathrm{End}(T^{(1,0)}\widetilde{M}_0)$ by $g^{TM}(\dot{R}^{\widetilde{L}} u, \overline{v}) = R^{\widetilde{L}}(u, \overline{v})$ for $u, v \in T^{(1,0)}\widetilde{M}_0$. We extend $\dot{R}^{\widetilde{L}}$ to $T\widetilde{M}_0 \otimes \mathbb{C} = T^{(1,0)}\widetilde{M}_0 \oplus T^{(0,1)}\widetilde{M}_0$ by setting $\dot{R}^{\widetilde{L}}\overline{v} = -\overline{\dot{R}^{\widetilde{L}} v}$. Then $\sqrt{-1}\dot{R}_x^{\widetilde{L}}$ induces an anti-symmetric endomorphism of $T\widetilde{M}_0$. Then from the formula for the heat kernel of a harmonic oscillator (see [20, (E.2.4), (E.2.5)] for instance), we find:

$$
\begin{aligned}
e^{-u\widetilde{\mathscr{L}_0}}(g^{-1}\widetilde{Z}, \widetilde{Z}) &= \frac{1}{(2\pi)^n} \frac{\det(\dot{R}_0^{\widetilde{L}}) e^{u\widetilde{\omega}_{d,0}}}{\det\left(1 - \exp(-u\dot{R}_0^{\widetilde{L}})\right)} \otimes \mathrm{Id}_{\widetilde{E}_0}\\
&\quad \times \exp\left\{ -\left\langle \frac{\dot{R}_0^{\widetilde{L}}/2}{\mathrm{th}(u\dot{R}_0^{\widetilde{L}}/2)} \widetilde{Z}, \widetilde{Z} \right\rangle + \left\langle \frac{\dot{R}_0^{\widetilde{L}}/2}{\mathrm{sh}(u\dot{R}_0^{\widetilde{L}}/2)} e^{u\dot{R}_0^{\widetilde{L}}/2} g^{-1}\widetilde{Z}, \widetilde{Z} \right\rangle \right\}.
\end{aligned}
\tag{49}
$$

Here, we use the convention that if an eigenvalue of $\dot{R}_0^{\widetilde{L}}$ is zero, then its contribution to the above term is $\widetilde{v} \mapsto \frac{1}{2\pi u} e^{-\frac{1}{2u}|g^{-1}\widetilde{v}-\widetilde{v}|^2}$.

We are now able to prove Theorems 3 and 4.

Proof (*of Theorem* 3) If x_0 is in M_{reg}, we have $G_{x_0} = \{1\}$ and the objects with or without tildas coincides. Thus, from (42), (43), and (48), (49) applied at $\widetilde{Z} = 0$, we get Theorem 3. □

Proof (*of Theorem* 4) We will use here the notations given in the introduction of this paper (before the statement of Theorem 4). In Particular, for $g \in G_{x_0}$, we have a decomposition $\widetilde{Z} = (\widetilde{Z}_{1,g}, \widetilde{Z}_{2,g})$ where $\widetilde{Z}_{1,g}$ is in the fixed-point set of g and $\widetilde{Z}_{2,g}$ is in the normal bundle of this set.

Let us fix $g \in G_{x_0}$. The idea is to apply the results of this Sect. 3 but replacing the base-point $0 \in \widetilde{U}_{x_0}$ by $\widetilde{Z}_{1,g}$. In order to stress on the dependence on $\widetilde{Z}_{1,g}$, we will add subscript to the various objects introduced above but defined with the base-point $\widetilde{Z}_{1,g}$, e.g., $\kappa_{\widetilde{Z}_{1,g}}$, $\widetilde{\mathscr{L}}_{0,\widetilde{Z}_{1,g}}$, etc... We can then make (48) more precise: observe that there is $c_0 > 0$ such that for each $g \in G_{x_0}$, $|\widetilde{Z}_{2,g} - g^{-1}\widetilde{Z}_{2,g}| \geq c_0|\widetilde{Z}_{2,g}|$, and thus for $m, \ell \in \mathbb{N}$ and $|\alpha'| \leq \ell$ we have some constants $c, C > 0$ such that

$$\begin{aligned}\sup_{|\alpha|\leq m} \Bigg| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}_{1,g}^{\alpha}} \frac{\partial^{|\alpha'|}}{\partial \widetilde{Z}_{2,g}^{\alpha'}} \Bigg(& p^{-n} e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(g^{-1}\widetilde{Z}, \widetilde{Z}) \\ & - e^{-u\widetilde{\mathscr{L}}_{0,\widetilde{Z}_{1,g}}}(\sqrt{p}g^{-1}\widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g})\widetilde{\kappa}^{-1}_{\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})\Bigg)\Bigg| \\ & \leq Cp^{\frac{\ell-1}{2}}\left(1+\sqrt{p}|\widetilde{Z}_{2,g}|\right)^N e^{-cp|\widetilde{Z}_{2,g}|^2}. \end{aligned} \tag{50}$$

In particular, we find that if $g = 1$ then $\widetilde{Z} = \widetilde{Z}_{1,g}$ and $\widetilde{Z}_{2,g} = 0$ and thus

$$\sup_{|\alpha|\leq m} \left| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^{\alpha}} \left(p^{-n} e^{-\frac{u}{p}\widetilde{L}_{p,x_0}}(\widetilde{Z}, \widetilde{Z}) - e^{-u\widetilde{\mathscr{L}}_{0,\widetilde{Z}}}(0,0)\right)\right| \leq Cp^{-1/2}. \tag{51}$$

Note that the image in the quotient of the union $\cup_{g\neq 1}\widetilde{U}^g_{x_0}$ is precisely $M_{sing} \cap U_{x_0}$. In particular, if Z is the image of $\widetilde{Z}$, then we have $|\widetilde{Z}_{2,g}| \geq d(Z, M_{sing})$ for $g \in G_{x_0} \setminus \{1\}$. From this remark and equations (42), (43), (50) and (51) we find that

$$\begin{aligned}\sup_{|\alpha|\leq \ell} \Bigg| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^{\alpha}} \Bigg(& p^{-n} e^{-\frac{u}{p}\Box_p}(\widetilde{Z}, \widetilde{Z}) - e^{-u\widetilde{\mathscr{L}}_{0,\widetilde{Z}}}(0,0) \\ & - \sum_{\substack{g\in G_{x_0} \\ g\neq 1}} (g,1).e^{-u\widetilde{\mathscr{L}}_{0,\widetilde{Z}_{1,g}}}(\sqrt{p}g^{-1}\widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g})\widetilde{\kappa}^{-1}_{\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})\Bigg)\Bigg| \\ & \leq Cp^{-1/2} + Cp^{\frac{\ell-1}{2}}(1+\sqrt{p}\, d(Z, M_{sing}))^N e^{-cp\, d(Z,M_{sing})^2}. \end{aligned} \tag{52}$$

To conclude, we get Theorem 4 thanks to the definition of $e^{i\theta_g}$ and g^E, and (11), (14), (49) and (52), noticing that $\dot{R}^{\widetilde{L}}$ and $\widetilde{\omega}_d$ coincide with the invariant lifts of $\dot{R}^L$ and ω_d. □

4 Proof of the inequalities

In this section, we will first prove Theorem 2, and then show how to use it in conjunction with the convergence of the heat kernel proved in Sect. 3 to get Theorem 1. The method is inspired by [3] (see also [20, Section 1.7]).

Proof (*of Theorem* 2) If λ is an eigenvalue of $\square_p$ acting on $\Omega^{0,j}(M, L^p \otimes E)$, we denote by F_j^λ the corresponding finite-dimensional eigenspace. As $\bar\partial^{L^p\otimes E}$ and $\bar\partial^{L^p\otimes E,*}$ commute with $\square_p$, we deduce that

$$\bar\partial^{L^p\otimes E}(F_j^\lambda) \subset F_{j+1}^\lambda \quad \text{and} \quad \bar\partial^{L^p\otimes E,*}(F_j^\lambda) \subset F_{j-1}^\lambda. \tag{53}$$

As a consequence, we have a complex

$$0 \longrightarrow F_0^\lambda \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} F_1^\lambda \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} \cdots \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} F_n^\lambda \longrightarrow 0. \tag{54}$$

If $\lambda = 0$, we have $F_j^0 \simeq H^j(M, L^p \otimes E)$ by Theorem 6. If $\lambda > 0$, then the complex (54) is exact. Indeed, if $\bar\partial^{L^p\otimes E} s = 0$ and $s \in F_j^\lambda$, then

$$s = \lambda^{-1}\square_p s = \lambda^{-1}\bar\partial^{L^p\otimes E}\bar\partial^{L^p\otimes E,*}s \in \mathrm{Im}(\bar\partial^{L^p\otimes E}). \tag{55}$$

In particular, we get for $\lambda > 0$ and $0 \le q \le n$

$$\sum_{j=0}^{q}(-1)^{q-j}\dim\, F_j^\lambda = \dim\left(\bar\partial^{L^p\otimes E}(F_q^\lambda)\right) \ge 0, \tag{56}$$

with equality if $q = n$.

Now, by Theorem 6

$$\mathrm{Tr}\,|_{\Omega^j}[e^{-\frac{u}{p}\square_p}] = \dim(H^j(M, L^p\otimes E) + \sum_{\lambda>0} e^{-\frac{u}{p}\lambda}\dim\, F_j^\lambda. \tag{57}$$

Finally, (56) and (57) entail (9). □

We can now conclude.

Proof (*of Theorem* 1) Let $\{x_i\}_{1\le i\le m}$ be a finite set of points of M_{sing} such that the corresponding local charts $(G_{x_i}, \widetilde{U}_{x_i})$ (as in Lemma 1) with $\widetilde{U}_{x_i} \subset \mathbb{C}^n$ satisfy

$$B^{\widetilde{U}_{x_i}}(0, 2\varepsilon) \subset \widetilde{U}_{x_i} \quad \text{and} \quad M_{sing} \subset \bigcup_{i=1}^{m} W_i\,, \quad W_i := B^{\widetilde{U}_{x_i}}\left(0, \frac{\varepsilon}{4}\right)/G_{x_i}. \tag{58}$$

Here ε is as in Sect. 3. Let W_0 be an open neighborhood of the complementary of $\bigcup_{i=1}^m W_i$ which is relatively compact in M_{reg}. Let $\{\psi_k\}_{0\le k\le m}$ be a partition of the unity subordinated to $\{W_k\}_{0\le k\le m}$.

In the sequel, we denote by $\mathrm{Tr}_{\Lambda^{0,q}}$ the trace on either $\Lambda^{0,q}(T^*M)\otimes L^p\otimes E$ or $\Lambda^{0,q}(T^*M)$. For $0 \le q \le n$, we have

$$\begin{aligned}\mathrm{Tr}_q\left[e^{-\frac{u}{p}\square_p}\right] &= \int_M \mathrm{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x,x)]dv_M(x)\\ &= \sum_{k=0}^{m}\int_M \psi_k(x)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x,x)]dv_M(x).\end{aligned} \tag{59}$$

From Theorem 3, we know that for $p \to \infty$

$$\begin{aligned}&p^{-n}\int_M \psi_0(x)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x,x)]dv_M(x)\\ &\quad= \frac{\mathrm{rk}(E)}{(2\pi)^n}\int_M \psi_0(x)\frac{\det(\dot{R}_x^L)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\left(1-\exp(-u\dot{R}_x^L)\right)}dv_M(x) + o(1).\end{aligned} \tag{60}$$

For $1 \leq k \leq m$, we know from Theorem 4 that for $p \to \infty$

$$
\begin{aligned}
&p^{-n} \int_M \psi_k(x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x,x)]dv_M(x) \\
&= \sum_{\substack{g \in G_{x_k} \\ g \neq 1}} \frac{1}{|G_{x_k}|} \int_{\{|\widetilde{Z}| \leq \frac{\varepsilon}{4}\}} \psi_k(\widetilde{Z}) e^{ip\theta_g} g^E(\widetilde{Z}_{1,g}) \mathcal{L}im_u(\widetilde{Z}_{1,g}) \mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \sqrt{p}\widetilde{Z}_{2,g}) dv_{\widetilde{TM}}(\widetilde{Z}) \\
&\quad + \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \psi_k(x) \frac{\det(\dot{R}^L_x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\left(1 - \exp(-u\dot{R}^L_x)\right)} dv_M(x) + o(1).
\end{aligned}
\tag{61}
$$

However, for $g \neq 1$, observe that

$$
\begin{aligned}
&\int_{\{|\widetilde{Z}| \leq \frac{\varepsilon}{4}\}} \psi_k(\widetilde{Z}) e^{ip\theta_g} g^E(\widetilde{Z}_{1,g}) \mathcal{L}im_u(\widetilde{Z}_{1,g}) \mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \sqrt{p}\widetilde{Z}_{2,g}) dv_{\widetilde{TM}}(\widetilde{Z}) \\
&\quad = p^{-\frac{\dim N_{x_k,g}}{2}} \int_{A(p,\varepsilon)} \psi_k\left(\left(\widetilde{Z}_{1,g}, \frac{\widetilde{Z}'_{2,g}}{\sqrt{p}}\right)\right) e^{ip\theta_g} g^E(\widetilde{Z}_{1,g}) \\
&\qquad \times \mathcal{L}im_u(\widetilde{Z}_{1,g}) \mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \widetilde{Z}'_{2,g}) dv_{\widetilde{TM}}(\widetilde{Z}),
\end{aligned}
\tag{62}
$$

where $A(p,\varepsilon) = \{|\widetilde{Z}_{1,g}|^2 + \frac{1}{p}|\widetilde{Z}'_{2,g}|^2 \leq \frac{\varepsilon}{4}\}$. Now, as we have $\langle (1-g^{-1})\widetilde{Z}'_{2,g}, \widetilde{Z}'_{2,g}\rangle \geq c_1|\widetilde{Z}'_{2,g}|$ with $c_1 > 0$, we can see that $\mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \widetilde{Z}'_{2,g})$ is exponentially decaying as $|\widetilde{Z}'_{2,g}| \to \infty$. Thus, there is $C > 0$ such that

$$
\left| \int_{A(p,\varepsilon)} \psi_k\left(\left(\widetilde{Z}_{1,g}, \frac{\widetilde{Z}'_{2,g}}{\sqrt{p}}\right)\right) e^{ip\theta_g} g^E(\widetilde{Z}_{1,g}) \mathcal{L}im_u(\widetilde{Z}_{1,g}) \mathcal{E}_{g,\widetilde{Z}_{1,g}}(u, \widetilde{Z}'_{2,g}) dv_{\widetilde{TM}}(\widetilde{Z}) \right| \leq C.
\tag{63}
$$

as a consequence, since $\dim N_{x_k,g} > 0$ for $g \neq 1$, we deduce that all the integrals (62) for $1 \leq k \leq m$ are $o(1)$ as $p \to \infty$. Thus, at the end, (61) turns out to reduce to

$$
\begin{aligned}
&p^{-n} \int_M \psi_k(x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x,x)]dv_M(x) \\
&\quad = \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \psi_k(x) \frac{\det(\dot{R}^L_x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\left(1 - \exp(-u\dot{R}^L_x)\right)} dv_M(x) + o(1).
\end{aligned}
\tag{64}
$$

From (59), (60) and (64), we find that for $p \to \infty$,

$$
p^{-n} \operatorname{Tr}_q[e^{-\frac{u}{p}\square_p}] = \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \frac{\det(\dot{R}^L_x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\left(1 - \exp(-u\dot{R}^L_x)\right)} dv_M(x) + o(1).
\tag{65}
$$

On the other hand, for $x \in X$, let $\{\widetilde{w}_j\}_{1 \leq j \leq n}$ be an local orthonormal frame of $\widetilde{T^{1,0}M_{U_x}}$ such that $\dot{R}^L \widetilde{w}_j = a_j(\widetilde{Z})\widetilde{w}_j$ and let $\{\widetilde{w}^j\}_{1 \leq j \leq n}$ be its dual basis. Then we have the following formula on $\widetilde{U}_x$:

$$
\begin{aligned}
\omega_d(\widetilde{Z}) &= -\sum_{j=0}^n a_j(\widetilde{Z}) \overline{\widetilde{w}}^j \wedge i_{\overline{\widetilde{w}}_j}, \\
e^{u\omega_d(\widetilde{Z})} &= \prod_{j=0}^n \left(1 + (e^{-ua_j(\widetilde{Z})} - 1)\right) \overline{\widetilde{w}}^j \wedge i_{\overline{\widetilde{w}}_j},
\end{aligned}
\tag{66}
$$

$$\frac{\det(\dot{R}^L_x)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\big(1-\exp(-u\dot{R}^L_x)\big)} = \left(\sum_{j_1<\cdots<j_q} e^{-u\sum_{k=1}^q a_{j_k}(x)}\right)\prod_{j=0}^n \frac{a_j(x)}{\big(1-e^{-ua_j(x)}\big)}.$$

In particular, the term in the integral in the right-hand side of (65) is uniformly bounded for $x \in M$ and $u > 0$, and moreover,

$$\lim_{u\to\infty} \frac{1}{(2\pi)^n}\frac{\det(\dot{R}^L_x)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\big(1-\exp(-u\dot{R}^L_x)\big)} = (-1)^q \mathbf{1}_{M(q)}(x)\det(\dot{R}^L_x/2\pi), \tag{67}$$

where $\mathbf{1}_{M(q)}$ denotes the indicator function of $M(q)$.

From Theorem 2 and (65), we have for $0 \le q \le n$ and any $u > 0$

$$\begin{aligned}&\limsup_{p\to\infty} p^{-n}\sum_{j=0}^q(-1)^{q-j}\dim H^j(M,{}^p\otimes E)\\ &\quad\le \frac{\mathrm{rk}(E)}{(2\pi)^n}\int_M\sum_{j=0}^q \frac{\det(\dot{R}^L_x)\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det\big(1-\exp(-u\dot{R}^L_x)\big)}dv_M(x).\end{aligned} \tag{68}$$

This, together with (67) and dominated convergence for $u \to \infty$, gives

$$\limsup_{p\to\infty} p^{-n}\sum_{j=0}^q(-1)^{q-j}\dim H^j(M,{}^p\otimes E) \le (-1)^q\mathrm{rk}(E)\int_{M(\le q)}\det(\dot{R}^L_x/2\pi)dv_M(x). \tag{69}$$

Finally, we have

$$\det(\dot{R}^L_x/2\pi)dv_M(x) = \frac{1}{n!}\left(\frac{\sqrt{-1}}{2\pi}R^L\right)^n, \tag{70}$$

which conclude the proof of Theorem 1. □

5 Moishezon orbifolds

In this section, we introduce the concept of Moishezon orbifolds, in a similar way as in the smooth case, and we give a criterion for a compact connected orbifold to be Moishezon. We thus prove that the Siu's [23,24] and Demailly's [10] answers to the Grauert–Riemenschneider conjecture [14] are still valid in the orbifold case. We follow here the same lines as the presentation of these results given in [20, Section 2.2].

In all this section, we consider a compact connected complex orbifold M, with set of orbifold charts $\mathcal{U}$. Also, as we mentioned before, we may assume without loss of generality that all the vector bundles in this section are proper.

5.1 Definition of Moishezon orbifolds

Let $\mathscr{S}_M \subset \mathscr{O}_M$ be the subsheaf such that for every open U, $\mathscr{S}_M(U)$ consists of the functions $f \in \mathscr{O}_M(U)$ which do not vanish identically on any connected components of U.

Definition 8 Let $\mathscr{M}_M$ be the sheaf associated with the pre-sheaf $U \mapsto \mathscr{S}_M(U)^{-1}\mathscr{O}_M(U)$. The section of $\mathscr{M}_M$ over an open set U are called the *meromorphic functions* on U.

By definition, $f \in \mathscr{M}_M(U)$ can be written in a (connected) neighborhood $V \in \mathcal{U}$ of any point as $f = g/h$ with $g \in \mathscr{O}_M(V)$ and $h \in \mathscr{S}_M(V)$. On such a neighborhood, we thus have two G_V-invariant holomorphic functions $\tilde{g}$, $\tilde{h}$ on $\widetilde{V}$, with $\tilde{h} \not\equiv 0$, such that the covering $\tilde{f}$ of f is given by $\tilde{f} = \tilde{g}/\tilde{h}$. Note that it is *a priori* stronger than asking that f is covered by a G_V-invariant meromorphic function on $\widetilde{V}$.

Definition 9 We say that $f_1, \ldots, f_k \in \mathscr{M}_M(M)$ are *algebraically independent* if for any polynomial $P \in \mathbb{C}[z_1, \ldots, z_k]$, $P(f_1, \ldots, f_k) = 0$ implies $P = 0$.

The *transcendence degree* of $\mathscr{M}_M(M)$ over $\mathbb{C}$ is the maximal number of algebraically independent meromorphic functions on M. We denoted it by $a(M)$.

By a theorem of Cartan and Serre [7] (see [8]), we know that $(M, \mathscr{O}_M)$ is an complex space, and thus we have $a(M) \leq \dim M$.

Definition 10 The compact connected orbifold M is called a *Moishezon orbifold* if it possesses $\dim M$ algebraically independent meromorphic functions, that is if $a(M) = \dim M$.

5.2 The Koidaira map and big line bundles

Let L be a Hermitian orbifold line bundle on M. Let $\mathbb{P}^*(H^0(M, L))$ be the Grassmannian of hyperplans of $H^0(M, L)$ (which can be identified to $\mathbb{P}(H^0(M, L)^*)$).

As for smooth compact manifold, we define the *base point locus* $\mathrm{BL}_{H^0(M,L)}$ of $H^0(M, L)$ as

$$\mathrm{BL}_{H^0(M,L)} = \{x \in M \ : \ \forall s \in H^0(M, L),\ s(x) = 0\}, \tag{71}$$

and we define the *Kodaira map* by

$$\begin{aligned} \Phi_L \colon M\backslash \mathrm{BL}_{H^0(M,L)} &\to \mathbb{P}^*(H^0(M, L)) \\ x &\mapsto \{s \in H^0(M, L) \ : \ s(x) = 0\}. \end{aligned} \tag{72}$$

If we chose a basis $(s_1, \ldots, s_d)$ of $H^0(M, L)$, then we have the following description of Φ_L: we use this basis to identify $H^0(M, L) \simeq \mathbb{C}^d$ and $\mathbb{P}^*(H^0(M, L)) \simeq \mathbb{P}^{d-1}(\mathbb{C})$, and under this identification we have

$$\Phi_L(x) = [s_1(x), \ldots, s_d(x)]. \tag{73}$$

This notation is slightly abusive because $s_i(x) \in L_x$. What is meant in (73) is that if we chose a local basis e_L of L we can write $s_i(x) = f_i(x)e_L$ and then $\Phi_L(x) = [f_1(x), \ldots, f_d(x)]$, but this element of $\mathbb{P}^{d-1}(\mathbb{C})$ does not depend on e_L, which justifies the abuse of notation. In fact, on $\{s_i \neq 0\}$ we have $\Phi_L(x) = [s_1(x)/s_i(x), \ldots, s_d(x)/s_i(x)]$.

In a local chart $(G_x, \widetilde{U}_x)$ near $x \in M$, the sections s_i are covered by G_x-equivariant holomorphic sections $\tilde{s}_i \colon \widetilde{U}_x \to \widetilde{L}_{U_x}$ and the G_x-invariant cover $\widetilde{\Phi}_L \colon \widetilde{U}_x \to \mathbb{P}^{d-1}(\mathbb{C})$ of Φ_L is given by $\widetilde{\Phi}_L(x) = [\tilde{s}_1(x), \ldots, \tilde{s}_d(x)]$. From this, we see that Φ_L is an orbifold holomorphic function on M.

Definition 11 For $p \in \mathbb{N}^*$, let $\varrho_p = \max\{\mathrm{rk}_x\, \Phi_{L^p}\}_{x \in M_{reg}\backslash \mathrm{BL}_{H^0(M,L^p)}}$.

The *Kodaira–Itaka* dimension $\kappa(L)$ of L is $\kappa(L) = \max\{\varrho_p\}_{p\in\mathbb{N}^*}$, and the line bundle L is called *big* if $\kappa(L) = \dim M$.

We have the following characterization of big line bundles.

Theorem 8 *Let M be a compact connected complex orbifold of dimension n and let L be a holomorphic orbifold line bundle on M. Then L is big if and only if*

$$\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) > 0. \tag{74}$$

For the proof this theorem, we will need a version of Siegel's lemma [22] (see also [20, Lemma 2.2.6]) for orbifolds. We will prove it first, drawing our inspiration from the approach of Andreotti [1].

Let $x \in M$ and $r > 0$ small. Let $(G_y, \widetilde{U}_y)$ be an orbifold chart as in Lemma 1, such that $x \in U_y$. For a pullback $\tilde{x} \in \widetilde{U}_y$ of x, we denote by $\widetilde{P}(\tilde{x}, r) = \{z \in \widetilde{U}_y : |z_i - \tilde{x}_i| \leq r \text{ for } 1 \leq i \leq n\}$ the polydisc in $\widetilde{U}_y$ of center $\tilde{x}$ and of radius r and by $P(x, r)$ its image in M. The Shilov boundary $\widetilde{S}(\tilde{x}, r)$ of $\widetilde{P}(\tilde{x}, r)$ is defined by $\widetilde{S}(\tilde{x}, r) = \{z \in \widetilde{U}_x : |z_i - \tilde{x}_i| = r \text{ for } 1 \leq i \leq n\}$, and its image in M is denoted by $S(x, r)$.

We warn the reader that the notations $P(x, r)$ and $S(x, r)$ can be somewhat misleading because these sets depend on the choice of $(G_y, \widetilde{U}_y)$ and of $\tilde{x}$, but it will not matter in the following proofs.

In the sequel, if $(G, \widetilde{U})$ is a chart and $X \subset \widetilde{U}$, we will set

$$[X] = \bigcup_{g\in G} g.X. \tag{75}$$

Lemma 2 *Let M be a compact connected complex orbifold of dimension n and let L be a holomorphic orbifold line bundle on M. Consider points $x_1, \dots, x_m$ in M_{reg} and positive numbers $r_1, \dots, r_m$. Using the above notations, we assume that $\widetilde{L}_{U_{y_i}}|_{[\widetilde{P}(\tilde{x}_i, 2r_i)]}$ is (equivariantly) trivial for $1 \leq i \leq m$ and $M \subset \cup_{i=1}^m P(x_i, e^{-1} r_i)$. Then there exists $k \in \mathbb{N}$ such that if $s \in H^0(M, L)$ vanishes at each x_i up to order k, then $s = 0$. In particular,* $\dim H^0(M, L) \leq m\binom{n+k}{k}$.

Proof We first fix a trivialization of $\widetilde{L}_{U_{y_i}}$ over $\big[\widetilde{P}(\tilde{x}_i, 2r_i)\big]$ for each i.

For $1 \leq i, j \leq m$, let $K_{ij} := \overline{P(x_i, r_i)} \cap \overline{P(x_j, r_j)}$. We can cover K_{ij} by finitely many disjoint orbifold charts $(G_{V_{ij}^\alpha}, \widetilde{V}_{ij}^\alpha)$, $1 \leq \alpha \leq A_{ij}$, with $V_{ij}^\alpha \subset U_{y_i} \cap U_{y_j}$. Let $\widetilde{K}_{ij}^\alpha$ be the pre-image of $K_{ij} \cap V_{ij}^\alpha$ in $\widetilde{V}_{ij}^\alpha$.

For each α, we choose an equivariant embedding $\varphi_{ij\to i}^\alpha \in \Phi_{V_{ij}^\alpha, U_{y_i}}$ (resp. $\varphi_{ij\to j}^\alpha \in \Phi_{V_{ij}^\alpha, U_{y_j}}$). Then the image of $\widetilde{K}_{ij}^\alpha$ is contained in $[\overline{\widetilde{P}(\tilde{x}_i, r_i)}]$ (resp. $[\overline{\widetilde{P}(\tilde{x}_j, r_j)}]$). This also induces a unique choice of compatible isomorphisms of $G_{V_{ij}^\alpha}$-equivariant bundles $\varphi_{ij\to i}^{\alpha,L} \in \Phi_{V_{ij}^\alpha, U_{y_i}}^L$ and $\varphi_{ij\to j}^{\alpha,L} \in \Phi_{V_{ij}^\alpha, U_{y_j}}^L$.

Then we can trivialize $\widetilde{L}_{V_{ij}^\alpha}$ either by composing the isomorphism $\varphi_{ij\to i}^{\alpha,L}$ with the trivialization on $\big[\widetilde{P}(\tilde{x}_i, 2r_i)\big]$, or by doing the same but replacing i by j. This gives rise to $G_{V_{ij}^\alpha}$-invariant holomorphic transition functions:

$$\tilde{c}_{ij}^\alpha : \widetilde{V}_{ij}^\alpha \to \mathbb{C}^*. \tag{76}$$

Set

$$C(L) = \sup\{|\tilde{c}_{ij}^\alpha(\tilde{x})|\}_{\tilde{x}\in\widetilde{K}_{ij}^\alpha,\, 1\leq\alpha\leq A_{ij},\, 1\leq i,j\leq m}. \tag{77}$$

As $\tilde{c}_{ij}^\alpha = (\tilde{c}_{ji}^\alpha)^{-1}$, we have $C(L) \geq 1$.

Set $k = \lfloor \log C(L) \rfloor + 1 \in \mathbb{N}^*$, and consider $s \in H^0(M, L)$ vanishing at each x_i up to order k. In the given trivialization, s is covered for each i by a G_{y_i}-equivariant function $\tilde{s}_i : \left[\widetilde{P}(x_i, 2r_i)\right] \to \mathbb{C}$. Set

$$\|s\| = \sup\{|\tilde{s}_i(\tilde{x})|\}_{\tilde{x} \in \left[\overline{\widetilde{P}(x_i, r_i)}\right],\, 1 \le i \le m} = \sup\{|\tilde{s}_i(\tilde{x})|\}_{\tilde{x} \in \overline{\widetilde{P}(x_i, r_i)},\, 1 \le i \le m}. \tag{78}$$

The last identity holds because s_i is equivariant and a finite group acting linearly on $\mathbb{C}$ acts by isometries.

It is well-know that a holomorphic function f on a neighborhood of a polydisc $P \subset \mathbb{C}^n$ attains its maximum on $\overline{P}$ at a point of the Shilov boundary of P. Thus, there exist $q \in \{1, \dots, m\}$ and $\tilde{t}_q \in \widetilde{S}(\tilde{x}_q, r_q)$ so that $|\tilde{s}_q(\tilde{t}_q)| = \|s\|$. We can find $j \neq q$ such that t, the image of $\tilde{t}_q$ in M, is in $P(x_j, e^{-1} r_j)$, and in particular $t \in K_{qj}$.

Let α be such that $t \in K_{qj} \cap V^{\alpha}_{qj}$, and let $\tilde{t}^{\alpha}_{qj}$ be in the pre-image of t in $\widetilde{V}^{\alpha}_{qj}$. Let $g \in G_{y_q}$ be such that $\tilde{t}_q = g \varphi^{\alpha}_{qj \to q}(\tilde{t}^{\alpha}_{qj})$. Set $\tilde{t}_j = \varphi^{\alpha}_{qj \to j}(\tilde{t}^{\alpha}_{qj})$, then, by the definition of $\tilde{c}^{\alpha}_{qj}$, we know that $\tilde{s}_q(g^{-1}\tilde{t}_q) = \tilde{c}^{\alpha}_{qj}(\tilde{t}^{\alpha}_{qj})\tilde{s}_j(\tilde{t}_j)$. Hence,

$$\|s\| = |\tilde{s}_q(\tilde{t}_q)| = |\tilde{s}_q(g^{-1}\tilde{t}_q)| = |\tilde{c}^{\alpha}_{qj}(\tilde{t}^{\alpha}_{qj})\tilde{s}_j(\tilde{t}_j)| \le C(L)|\tilde{s}_j(\tilde{t}_j)|. \tag{79}$$

Now, let $h \in G_{y_j}$ be such that $h\tilde{t}_j \in \widetilde{P}(\tilde{x}_j, e^{-1} r_j)$. Applying the Schwartz inequality (see for instance [20, Problem 2.3]) to $\tilde{s}_j$ on $\widetilde{P}(\tilde{x}_j, r_j)$ we get

$$|\tilde{s}_j(\tilde{t}_j)| = |\tilde{s}_j(h\tilde{t}_j)| \le \|s\|.|h\tilde{t}_j|_0^k.r_j^{-k} \quad \text{where} \quad |(z_1, \dots, z_k)|_0 = \sup\{|z_k|\}_{1 \le k \le n}. \tag{80}$$

Hence, as $|h\tilde{t}_j|_0 \le e^{-1} r_j$, we conclude:

$$\|s\| \le \|s\| C(L) e^{-k}, \tag{81}$$

which implies that $s = 0$ by the definition of k.

Finally, this proves that the map from $H^0(M, L)$ to the product of m copies of the space $\mathbb{C}_k[X_1, \dots, X_n]$ of polynomial in n variables and of degree $\le k$, which associate to each section its k-jets at $x_1, \dots, x_m$, is injective. As $\dim \mathbb{C}_k[X_1, \dots, X_n] = \binom{n+k}{k}$, Lemma 2 is proved. □

Theorem 9 (Siegel's lemma for orbifolds) *Let M be a compact connected complex orbifold and let L be a holomorphic orbifold line bundle on M. Then there exists $C > 0$ such that for any $p \in \mathbb{N}^*$,*

$$\dim H^0(M, L^p) \le C p^{\varrho_p}. \tag{82}$$

Proof We modify the proof of Lemma 2 and we will use the same notations as there. The set of points where Φ_{L^p} has rank less than ϱ_p is a proper analytic set of M. As a consequence, the set $\{x \in M : \forall\, p \in \mathbb{N}^*,\ \mathrm{rk}_x \Phi_{L^p} = \varrho_p\}$ is dense in M and we can choose $(x_1, \dots, x_m) \in M_{reg}$ as in Lemma 2 such that Φ_{L^p} has rank ϱ_p at x_i for any $p \in \mathbb{N}^*$.

Since Φ_{L^p} has constant rank near x_i, there exists a ϱ_p-dimensional submanifold $M_{p,i}$ in a neighborhood of x_i which is transversal at x_j to the fiber $\Phi_{L^p}^{-1}(\Phi_{L^p}(x_i))$. Now, if $s \in H^0(M, L^p)$ vanishes up to order $k_p = p(\lfloor \log C(L) \rfloor + 1)$ at each x_i along $M_{p,i}$, then it also does along $\Phi_{L^p}^{-1}(\Phi_{L^p}(x_i))$, and thus it vanishes up to order k_p at each x_i (on M). With the same reasoning as in Lemma 2 we get that $\|s\| \le \|s\| C(L^p) e^{-k_p}$, and since $C(L^p) = C(L)^p$, we find that $s = 0$.

Finally, as in Lemma 2, we have proved that the map associating to each $s \in H^0(M, L^p)$ its k-jets along $M_{p,i}$ for $1 \le i \le m$ is injective. In particular, $\dim H^0(M, L^p) \le m\binom{\varrho_p + k_p}{k_p}$, which implies Theorem 9. □

We can now turn back to the characterisation of big line bundles.

Proof (*of Theorem* 8) Clearly, by Theorem 9, we get that if $\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) > 0$ then L is big.

Conversely, if L is big, there are $m \in \mathbb{N}^*$ and $x_0 \in M_{reg} \backslash \mathrm{BL}_{H^0(M,L^m)}$ such that $\mathrm{rk}_{x_0} \Phi_{L^m} = n$. Thus, there are $s_0, \dots, s_n \in H^0(M, L^m)$ such that $s_0(x_0) \neq 0$ and $d(\frac{s_1}{s_0})_{x_0} \wedge \cdots \wedge d(\frac{s_n}{s_0})_{x_0} \neq 0$. Hence, $(\frac{s_1}{s_0}(x), \dots, \frac{s_n}{s_0}(x))$ are local coordinates near x_0.

Therefor, if a polynomial P of degree p in n variables is such that $P(\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}) = 0$, then $P = 0$. In particular, if Q is a non-zero homogeneous polynomial of degree p in $n + 1$ variables, then $Q(s_0, \dots, s_n) \in H^0(M, L^{rm})$ is also non-zero. In deed if it was, then $P(X_1, \dots, X_n) := Q(1, X_1, \dots, X_n) \neq 0$ will satisfy

$$P\left(\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}\right) = \frac{1}{s_0^r} Q(s_0, \dots, s_n) = 0,$$

a contradiction. As the space of homogeneous polynomials of degree p in $n + 1$ variables has dimension $\binom{n+p}{p} \geq p^n/n!$, we deduce that $\dim H^0(M, L^{mp}) \geq p^n/n!$ and thus

$$\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) \geq 1/n! > 0.$$

Theorem 8 is proved. □

5.3 A criterion for Moishezon orbifolds

Lemma 3 *Let M be a compact connected complex orbifold. If M carries a big line bundle, then M is Moishezon.*

Proof If L is a big line bundle on M, there are $m \in \mathbb{N}^*$ and $x_0 \in M_{reg} \backslash \mathrm{BL}_{H^0(M,L^m)}$ such that $\mathrm{rk}_{x_0} \Phi_{L^m} = n$. Thus, there are $s_0, \dots, s_n \in H^0(M, L^m)$ such that $s_0(x_0) \neq 0$ and $d(\frac{s_1}{s_0})_{x_0} \wedge \cdots \wedge d(\frac{s_n}{s_0})_{x_0} \neq 0$. Hence, $(\frac{s_1}{s_0}(x), \dots, \frac{s_n}{s_0}(x))$ are local coordinates near x_0. Therefor, if a polynomial P in n variables is such that $P(\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}) = 0$, then $P = 0$. Thus, the meromorphic functions $\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}$ are algebraically independent, so that $a(M) \geq n$ and M is Moishezon. □

Remark 6 In the case of a regular compact connected complex manifold, it is in fact equivalent to be Moishezon and to carry a big line bundle (see for instance [20, Theorem 2.2.15]), but in the singular case, the proof cannot be directly adapted and to the knowledge of the author it is not known wether it is also an equivalence.

We can now prove the criterion stated in Theorem 5 in the introduction of this paper.

Proof (*of Theorem* 5) First, observe that if (L, h^L) is semi-positive, then $X(1) = \emptyset$ and if moreover (L, h^L) is positive at one point then $X(0) \neq \emptyset$ so that $\int_{M(\leq 1)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n = \int_{M(0)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n$ is positive. Thus *(i)* implies *(ii)*. We will now prove that *(ii)* implies that M is Moishezon.

If we apply Theorem 1 for $q = 1$, we find

$$\dim H^0(M, L^p) \geq \frac{p^n}{n!} \int_{M(\leq 1)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n + o(p^n), \tag{83}$$

and thus with the hypothesis (16) and Theorem 8 we find that L is big. By Lemma 3, we find that M is Moishezon. □

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