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Inégalités de Morse holomorphes G-invariantes et formes de torsion asymptotiques

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À ma famille en général
et à Julie en particulier

On sentait que le chef de cette barricade
était un géomètre ou un spectre.
Victor Hugo, *Les Misérables*

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Résumé

Résumé

Dans cette thèse, nous étudions certains aspects de la limite semi-classique en géométrie complexe. Soit M une variété complexe munie d'un fibré en droites holomorphe L et d'un fibré complexe E. Nous donnons ici les propriétés asymptotiques d'objets associés aux grandes puissances tensorielles de L tordues par E.

Dans le premier chapitre, M est compacte, L positif et E pas nécessairement holomorphe. Nous montrons l'annulation des $2j$ premiers termes du développement asymptotique diagonal du noyau de Bergman restreint aux $(0, 2j)$-formes, puis nous donnons une formule locale pour les coefficients dominants.

Dans le deuxième chapitre, M est compacte, E holomorphe et un groupe de Lie compact connexe agit sur M, L et E de façon compatible. Nous établissons des inégalités de Morse holomorphes analogues à celles de Demailly pour la partie invariante de la cohomologie de Dolbeault. Pour cela, nous définissons, sous une hypothèse naturelle, la réduction de M et nous donnons nos inégalités en termes de la courbure du fibré induit par L sur cette réduction.

Dans le troisième chapitre, E est holomorphe et M est l'espace total d'une fibration holomorphe de fibre compacte. On peut alors définir les formes de torsion analytique holomorphe associées à cette fibration et aux puissances de L tordues par E. Nous donnons d'abord une formule asymptotique pour ces formes, que nous généralisons ensuite dans le cas où les puissances de L sont remplacées par l'image directe des puissances d'un fibré en droites sur une variété plus grosse. Dans les deux cas, nous devons faire des hypothèses de positivité sur le fibré en droites. Ces résultats sont les versions en famille des résultats de Bismut-Vasserot [17, 18].

Mots-clefs

Quantification, limite semi-classique, développement asymptotique, noyau de Bergman, noyau de la chaleur, inégalités de Morse holomorphes, réduction, formes de torsion de Bismut-Köhler, opérateurs de Toeplitz.

G-invariant holomorphic Morse inequalities and asymptotic torsion forms

Abstract

In this thesis, we study some aspects of the semi-classical limit in complex geometry. Let M be a complex manifold, endowed with a holomorphic line bundle L and a complex bundle E. We give here the asymptotic properties of several objects associated with the high tensor powers of L, twisted by E.

In the first chapter, M is compact, L positive and E non necessarily holomorphic. We prove the cancellation of the first $2j$ terms in the diagonal asymptotic expansion of the restriction to the $(0, 2j)$-forms of the Bergman kernel. Then, we give a local formula for the leading coefficients.

In the second chapter, M is compact, E holomorphic and a connected compact Lie group acts on M, L and E in a compatible way. We establish asymptotic holomorphic Morse inequalities à la Demailly for the invariant part of the Dolbeault cohomology. To do so, we define the reduction of M under natural hypothesis and give our inequalities in terms of the curvature of the bundle induced by L on this reduction.

In the third chapter, E is holomorphic and M is the total space of a holomorphic fibration with compact fibers. We can then define the holomorphic analytic torsion forms associated with this fibration and the tensor powers of L, twisted by E. We first give an asymptotic formula for these forms. Secondly, we generalize this formula in the case where the powers of L are replaced by the direct image of powers of a line bundle on a bigger manifold. In both cases we have to make positivity assumptions on the line bundle. These results are the family versions of the results of Bismut-Vasserot [17, 18].

Keywords

Quantization, semi-classical limit, asymptotic expansion, Bergman kernel, heat kernel, holomorphic Morse inequalities, reduction, torsion forms of Bismut-Köhler, Toeplitz operators.

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Introduction

En géométrie symplectique, une variété *préquantifiée* est une variété symplectique compacte (M, ω) telle qu'il existe un fibré en droites complexes hermitien (L, h^L) muni d'une connexion hermitienne ∇^L vérifiant $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Dans le cadre de la géométrie kählerienne, M est de plus munie d'une structure complexe J et ω est une forme de Kähler. On demande alors de plus que L soit un fibré holomorphe. Dans ce cas, le choix naturel pour ∇^L est la connexion de Chern de (L, h^L). Considérons de plus un fibré holomorphe E sur M. Nous pouvons alors définir la *quantification* de M de la manière suivante. Soit $\bar{\partial}^{L\otimes E}$ l'opérateur de Dolbeault sur l'espace $\Omega^{0,\bullet}(M, L\otimes E)$ des formes de type $(0, \bullet)$ à valeurs dans $L \otimes E$ et $\square^{L\otimes E} = \bar{\partial}^{L\otimes E}\bar{\partial}^{L\otimes E,*} + \bar{\partial}^{L\otimes E,*}\bar{\partial}^{L\otimes E}$ le laplacien de Kodaira. La quantification de M est alors l'espace vectoriel virtuel de dimension finie

$$Q(M, L\otimes E) = \ker(\square^{L\otimes E}|_{\Omega^{0,\mathrm{pair}}}) - \ker(\square^{L\otimes E}|_{\Omega^{0,\mathrm{impair}}}). \tag{0.0.1}$$

Notons $H^j(M, L\otimes E)$ la cohomologie de Dolbeault en bidegré $(0, j)$ associée à $L \otimes E$, qui est aussi le j-ième groupe de cohomologie du faisceau des sections holomorphes de $L \otimes E$. D'après la théorie de Hodge, on a alors

$$Q(M, L\otimes E) = H^{\mathrm{pair}}(M, L\otimes E) - H^{\mathrm{impair}}(M, L\otimes E). \tag{0.0.2}$$

Dans ce contexte de quantification, la limite semi-classique de la physique correspond à remplacer le fibré L par ses puissances tensorielles $L^p := L^{\otimes p}$ avec $p \to +\infty$, le paramètre de Planck $\hbar$ étant alors $\hbar = \frac{1}{p}$. En réalité, grâce à la formule (0.0.2), on peut définir la quantification de M sous des hypothèses moins restrictives. Par exemple dans le premier chapitre de cette thèse, on ne supposera pas que E est holomorphe et dans le deuxième on ne supposera pas que L est positif.

Un certain nombre de résultats importants de géométrie algébrique ou complexe ont été obtenus en étudiant cette limite semi-classique. Un des exemples les plus connus est le théorème de plongement de Kodaira, qui affirme qu'une variété complexe compacte M est projective si elle admet un fibré en droites positif L. En effet, d'une part le théorème d'annulation de Kodaira assure que dans ce cas $H^j(M, L^p)$ est nul pour $j \geq 1$ et p assez grand, et d'autre part on sait grâce au théorème de Hirzebruch-Riemann-Roch asymptotique que L^p a beaucoup de sections holomorphes quand $p \to +\infty$. C'est ce qui permet de plonger M dans l'espace projectif des hyperplans de $H^0(M, L^p)$.

Le but de cette thèse est d'étudier le comportement asymptotique de certains objets associés à $L^p \otimes E$ sous la limite semi-classique $p \to +\infty$. Plus précisément, dans le premier chapitre, nous étudions le noyau de Bergman ; dans le deuxième chapitre, un analogue des inégalités de Morse holomorphe dans un cadre équivariant et, dans le dernier chapitre, les formes de torsions analytique holomorphe. Dans chacun des chapitres, nous faisons des hypothèses différentes sur les données géométriques en jeux.

Les trois chapitres composant cette thèse sont indépendants, et nous donnons ci-dessous un introduction plus détaillée à chacun d'entre eux. Insistons cependant sur le fait que le point de vue que nous adoptons tout au long de cette thèse est inspiré de la théorie de l'indice locale et en particulier des techniques de localisation analytique de Bismut-Lebeau. Les objets centraux de cette thèse sont le laplacien ainsi que le noyau de la chaleur et ses propriétés asymptotiques.

0.1 Les termes dominants dans l'asymptotique du noyau de Bergman en degrés supérieurs

Cette partie de ma thèse est un travail en commun avec Jialin Zhu.

Le noyau de Bergman d'une variété kählerienne, munie d'un fibré en droites positif L, est le noyau de Schwartz lisse de la projection sur le noyau du laplacian de Kodaira $\square^L = \bar{\partial}^L \bar{\partial}^{L,*} + \bar{\partial}^{L,*}\bar{\partial}^L$. La preuve de l'existence d'un développement asymptotique sur la diagonale du noyau de Bergman associé à la p-ième puissance tensorielle de L, quand $p \to +\infty$, et la forme du terme principal sont données dans [61], [24] et [69]. Les coefficients de ce développement contiennent de l'information sur la géométrie de la variété de base, et ont par conséquent été étudiés de près : les deuxième et troisième termes ont été calculés par Lu [40], X. Wang [64], L. Wang [63] et par Ma-Marinescu [49] à différents degrés de généralité (voir aussi l'étude récente [68]). L'asymptotique du noyau de Bergman joue un rôle important dans divers problèmes de géométrie kählerienne, voir par exemple [31] ou [32]. Le lecteur pourra se référer au livre [46] pour une étude complète du noyau de Bergman et de ses applications, ainsi qu'au panorama [43].

En fait, Dai-Liu-Ma ont établi le développement asympototique du noyau de Bergman dans le cas symplectique dans [26], en utilisant le noyau de chaleur (cf. également Ma-Marinescu [45]). Récemment, le cas symplectique a trouvé une application dans l'étude de la variation de structures de Hodge de fibrés vectoriels par Charbonneau et Stern dans [25]. Dans leur article, le noyau de Bergman est le noyau associé à un laplacien de type de Kodaira sur un fibré tordu $L \otimes E$, où E est un fibré vectoriel complexe (non nécessairement holomorphe). Á cause de cela, le noyau de Bergman ne se concentre plus en degré 0 (contrairement au cas kählerien), et le développement asymptotique de sa restriction aux $(0, 2j)$-formes est lié au degré de « non-holomorphicité » de E.

Dans cet article, nous allons montrer que le premier terme dans l'asymptotique de la restriction aux $(0, 2j)$-formes du noyau de Bergman est d'ordre $p^{\dim X - 2j}$ et nous le calculerons. Cela conduira à une version locale de [25, (1.3)], qui est le résultat technique principal de leur article, voir la remarque 0.1.6. Après cela, nous calculerons également le deuxième terme de cette asymptotique, ainsi que le troisième dans le cas où les deux premiers s'annulent.

Donnons maintenant plus de détails sur nos résultats. Soit (X, ω, J) une variété kählerienne compacte de dimension complexe n. Soit (L, h^L) un fibré en droites hermitien holomorphe sur X, et (E, h^E) un fibré vectoriel complexe hermitien. Munissons (L, h^L) de sa connexion de Chern (i.e. holomorphe et hermitien) ∇^L, et (E, h^E) d'une connexion hermitienne ∇^E, dont les courbures sont respectivement $R^L = (\nabla^L)^2$ et $R^E = (\nabla^E)^2$.

Mis à part dans le début de la section 1.2.1, nous supposerons toujours que (L, h^L, ∇^L) satisfait la *condition de pré-quantification* :

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.1.1}$$

Soit $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ la métrique riemannienne sur TX induite par ω et J. Cette métrique à son tour induit une métrique $h^{\Lambda^{0,\bullet}}$ sur $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$, voir la section 1.2.1.

Soit $\Omega^{0,\bullet}(X, L^p \otimes E) = \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes E)$ et $\bar{\partial}^{L^p\otimes E} : \Omega^{0,\bullet}(X, L^p \otimes E) \to \Omega^{0,\bullet+1}(X, L^p \otimes E)$ l'opérateur de Dolbeault induit par la partie de degré $(0,1)$ de ∇^E. Soit $\bar{\partial}^{L^p\otimes E,*}$ son adjoint par rapport au produit scalaire L^2. Posons (voir (1.2.6)) :

$$D_p = \sqrt{2}\Big(\bar{\partial}^{L^p\otimes E} + \bar{\partial}^{L^p\otimes E,*}\Big), \tag{0.1.2}$$

qui échange les formes de degré impair et celles de degré pair.

Définition 0.1.1. Soit

$$P_p \colon \Omega^{0,\bullet}(X, L^p \otimes E) \to \ker(D_p) \tag{0.1.3}$$

la projection orthogonale sur le noyau $\ker(D_p)$ de D_p. L'opérateur P_p est appelé la *projection de Bergman*. Il admet un noyau lisse par rapport à $dv_X(y)$, noté $P_p(x,y)$ et appelé le *noyau de Bergman*.

Remarque 0.1.2. Si E est holomorphe, alors par la théorie de Hodge et le théorème d'annulation de Kodaira (voir respectivement [46, Theorem 1.4.1] et [46, Theorem 1.5.6]), nous savons que, pour p assez grand, P_p est la projection orthogonale $\mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$. Ici, par [44, Theorem 1.1], nous savons juste que $\ker(D_p|_{\Omega^{0,\mathrm{impair}}(X,L^p\otimes E)}) = 0$ pour p assez grand, de sorte que $P_p \colon \Omega^{0,\mathrm{pair}}(X, L^p \otimes E) \to \ker(D_p)$. En particulier, $P_p(x,x) \in \mathscr{C}^\infty\left(X, \mathrm{End}(\Lambda^{0,\mathrm{pair}}(T^*X) \otimes E)\right)$.

D'après le théorème 1.2.3, D_p est un opérateur de Dirac, ce qui nous permet d'appliquer le résultat suivant :

Théorème 0.1.3 (Dai-Liu-Ma, [26, Thm. 1.1])**.** *Il existe $\boldsymbol{b}_r \in \mathscr{C}^\infty(M, \mathrm{End}(\Lambda^{0,\mathrm{pair}}(T^*X) \otimes E))$ tels que, pour tout $k \in \mathbb{N}$ et pour $p \to +\infty$:*

$$p^{-n} P_p(x,x) = \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} + O(p^{-k-1}), \tag{0.1.4}$$

c'est-à-dire que pour tout $k, l \in \mathbb{N}$, il existe une constante $C_{k,l} > 0$ telle que pout tout $p \in \mathbb{N}$,

$$\left| p^{-n} P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} \right|_{\mathscr{C}^l(X)} \leq C_{k,l} p^{-k-1}. \tag{0.1.5}$$

Ici $|\cdot|_{\mathscr{C}^l(X)}$ est la norme $\mathscr{C}^l$ pour la variable $x \in X$.

Pour simplifier les formules, nous noterons

$$\mathscr{R} = (R^E)^{0,2} \in \Omega^{0,2}(X, \mathrm{End}(E)) \tag{0.1.6}$$

la partie de degré $(0,2)$ de R^E (qui est nulle si E est holomorphe). Pour $1 \leq j \leq n$, soit

$$I_j \colon \Lambda^{0,\bullet}(T^*X) \otimes E \to \Lambda^{0,j}(T^*X) \otimes E \tag{0.1.7}$$

la projection orthogonale naturelle. Le résultat principal de cette partie est

Théorème 0.1.4. *Pour tout $k \in \mathbb{N}$, $k \geq 2j$, nous avons quand $p \to +\infty$:*

$$p^{-n} I_{2j} P_p(x,x) I_{2j} = \sum_{r=2j}^{k} I_{2j} \boldsymbol{b}_r(x) I_{2j}\, p^{-r} + O(p^{-k-1}), \tag{0.1.8}$$

et de plus,

$$I_{2j} \boldsymbol{b}_{2j}(x) I_{2j} = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} I_{2j} \left(\mathscr{R}_x^j\right) \left(\mathscr{R}_x^j\right)^* I_{2j}, \tag{0.1.9}$$

où $(\mathscr{R}_x^j)^*$ est le dual de $\mathscr{R}_x^j$ agissant sur $(\Lambda^{0,\bullet}(T^*X) \otimes E)_x$.

Le Théorème 0.1.4 conduit immédiatement au

Corollaire 0.1.5. *Uniformément en $x \in X$, nous avons lorsque $p \to +\infty$*

$$\operatorname{Tr}\left(\left(I_{2j} P_p I_{2j}\right)(x,x)\right) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \left\|\mathscr{R}_x^j\right\|^2 p^{n-2j} + O(p^{n-2j-1}). \tag{0.1.10}$$

Remarque 0.1.6. En intégrant (0.1.10) sur X, nous obtenons

$$\operatorname{Tr}\left(I_{2j} P_p I_{2j}\right) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \left\|\mathscr{R}^j\right\|_{L^2}^2 p^{n-2j} + O(p^{n-2j-1}), \tag{0.1.11}$$

qui est le résultat technique principal de Charbonneau et Stern [25, (1.3)]. Le corollaire 0.1.5 peut donc être considéré comme une version locale de [25, (1.3)]. La constante dans (0.1.11) diffère de celle dans [25] car nos conventions ne sont pas les mêmes que dans [25] (par exemple, ils choisissent $\omega = \sqrt{-1}R^L$, etc...).

Soit $R^E_\Lambda := -\sqrt{-1}\sum_i R^E(w_i, \overline{w}_i)$ pour $(\overline{w}_1, \dots, \overline{w}_n)$ un repère orthonormé de $T^{(0,1)}X$. Soit R^{TX} la courbure de la connexion de Levi-Civita ∇^{TX} de (X, g^{TX}). Pour $(e_1, \dots, e_{2n})$ un repère orthonormé de TX, soit $r^X = -\sum_{i,j}\langle R^{TX}(e_i,e_j)e_i, e_j\rangle$ la courbure scalaire de X.

Pour $j, k \in \mathbb{N}$ et $j \geq k$, nous définissons aussi $\mathrm{C}_j(k)$ par

$$\mathrm{C}_j(k) := \frac{1}{(4\pi)^j} \frac{1}{2^k k!} \frac{1}{\prod_{s=k+1}^j (2s+1)}, \tag{0.1.12}$$

avec la convention $\prod_{s\in\emptyset} = 1$.

Soit $\nabla^{\Lambda^{0,\bullet}}$ la connexion sur $\Lambda^{0,\bullet}(T^*X)$ induite par ∇^{TX}. Soit $\nabla^{\Lambda^{0,\bullet}\otimes E}$ la connexion sur $\Lambda^{0,\bullet}(T^*X) \otimes E$ induite par ∇^E et $\nabla^{\Lambda^{0,\bullet}}$, et soit $\Delta^{\Lambda^{0,\bullet}\otimes E}$ le laplacien associé. Pour les définitions précises, voir la section 1.2.1.

Pour tout opérateur A agissant sur un espace hermitien, nous noterons $\mathrm{Pos}[A]$ (resp. $\mathrm{Sym}[A]$) l'opérateur positif (resp. symétrique) associé à A :

$$\mathrm{Pos}[A] = AA^* \quad \text{and} \qquad \mathrm{Sym}[A] = A + A^*. \tag{0.1.13}$$

Enfin, pour simplifier les notations, nous définissons $\mathcal{T}_0(j)$, $\mathcal{T}_1(j)$, $\mathcal{T}_2(j)$ et $\mathcal{T}_3(j)$ de la manière suivante :

- $\mathcal{T}_0(0) = 0$, et pour $j \geq 1$,

$$\mathcal{T}_0(j) = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^n \sum_{k=0}^{j-1} I_{2j}\Big(\mathrm{C}_j(j) - \mathrm{C}_j(k)\Big)\mathscr{R}_x^{j-k-1}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\mathscr{R}_x^k I_0. \tag{0.1.14}$$

- $\mathcal{T}_1(0) = \mathcal{T}_1(1) = 0$, et pour $j \geq 2$,

$$\begin{aligned}\mathcal{T}_1(j) = \frac{I_{2j}}{2\pi}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\Big\{&\Big(\mathrm{C}_j(j) - \mathrm{C}_j(q+1)\Big)\mathscr{R}_x^{j-(q+2)}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\mathscr{R}_x^{q-m}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i}\mathscr{R}.)(x)\mathscr{R}_x^m \\ &+ \mathrm{C}_j(m)\left[\prod_{s=q+2}^{j}\Big(1+\frac{1}{2s}\Big) - 1\right]\mathscr{R}_x^{j-(q+2)}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i}\mathscr{R}.)(x)\mathscr{R}_x^{q-m}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\mathscr{R}_x^m\Big\} I_0,\end{aligned} \tag{0.1.15}$$

- $\mathcal{T}_2(0) = 0$, et pour $j \geq 1$,

$$\mathcal{T}_2(j) = \frac{1}{4\pi} I_{2j} \sum_{k=0}^{j-1} \left\{ \Big(\mathrm{C}_j(j) - \mathrm{C}_j(k) \Big) \mathscr{R}_x^{j-(k+1)} (\Delta^{\Lambda^{0,\bullet}\otimes E} \mathscr{R}.)(x) \mathscr{R}_x^k \right\} I_0, \tag{0.1.16}$$

- pour $j \geq 0$,

$$\mathcal{T}_3(j) = I_{2j} \sum_{k=0}^{j} \mathscr{R}_x^{j-k} \left[\frac{1}{6} \left(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)} \right) r_x^X - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)} \sqrt{-1} R^E_{\Lambda,x} \right] \mathscr{R}_x^k I_0. \tag{0.1.17}$$

Le deuxième objectif de cette partie est de calculer le deuxième terme du développement (0.1.8).

Théorème 0.1.7. *Le terme $I_{2j}\boldsymbol{b}_{2j+1}(x) I_{2j}$ peut se décomposer en une somme de quatre termes comme suit :*

$$I_{2j}\boldsymbol{b}_{2j+1}(x) I_{2j} = \mathrm{Pos}[\mathcal{T}_0(j)] + \mathrm{C}_j(j)\mathrm{Sym}\left[(\mathcal{T}_1(j) + \mathcal{T}_2(j) + \mathcal{T}_3(j))(\mathscr{R}_x^j)^* I_{2j} \right]. \tag{0.1.18}$$

Par exemple, pour $j = 1$, en utilisant $(R^E_\Lambda)^* = R^E_\Lambda$, nous trouvons

$$\begin{aligned} 128\pi^3 I_2 \boldsymbol{b}_3(x) I_2 = {} & \frac{1}{9}\mathrm{Pos}\left[I_2 \sum_{i=0}^{n} (\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i} \mathscr{R}.)(x) I_0 \right] + \frac{1}{6}\mathrm{Sym}\left[I_2 (\Delta^{\Lambda^{0,\bullet}\otimes E} \mathscr{R}.)(x) \mathscr{R}_x^* I_2 \right] \\ & - \frac{\sqrt{-1}}{6} I_2 (R^E_\Lambda \mathscr{R}_x \mathscr{R}_x^* + \mathscr{R}_x \mathscr{R}_x^* R^E_\Lambda) I_2 - \frac{2\sqrt{-1}}{3} I_2 \mathscr{R}_x R^E_\Lambda \mathscr{R}_x^* I_2 - \frac{r_x^X}{4} I_2 \mathscr{R}_x \mathscr{R}_x^* I_2. \end{aligned} \tag{0.1.19}$$

Le dernier objectif de cette partie est de calculer le troisième terme du développement (0.1.8), sous l'hypothèse d'annulation des deux premiers.

Théorème 0.1.8. *Soit $j \in [\![1, n]\!]$. Si*

$$I_{2j}\boldsymbol{b}_{2j}(x) I_{2j} = I_{2j}\boldsymbol{b}_{2j+1}(x) I_{2j} = 0, \tag{0.1.20}$$

alors $\mathcal{T}_3$ vaut

$$\mathcal{T}_3'(j) := -\sqrt{-1} I_{2j} \sum_{k=0}^{j} \frac{\mathrm{C}_j(k)}{4\pi(2k+1)} \mathscr{R}_x^{j-k} R^E_{\Lambda,x} \mathscr{R}_x^k I_0, \tag{0.1.21}$$

et

$$I_{2j}\boldsymbol{b}_{2j+2}(x) I_{2j} = \mathrm{Pos}[\mathcal{T}_1(j) + \mathcal{T}_2(j) + \mathcal{T}_3'(j)]. \tag{0.1.22}$$

Les théorèmes 0.1.4, 0.1.7 et 0.1.8 impliquent le résultat suivant, qui donne des conditions sur la courbure de E pour que le noyau de Bergman se concentre plus rapidement en degré 0 que dans le cas général.

Corollaire 0.1.9. *On a l'équivalence*

$$I_{2j} P_p(x,x) I_{2j} = O(p^{n-2j-3}) \iff \begin{cases} \mathscr{R}_x^j = 0, \\ \mathcal{T}_0(j) = 0, \\ \mathcal{T}_1(j) + \mathcal{T}_2(j) + \mathcal{T}_3'(j) = 0. \end{cases} \tag{0.1.23}$$

0.2 Inégalités de Morse holomorphes G-invariantes

La théorie de Morse a pour but de comprendre l'information topologique portée par les fonctions de Morse sur une variété et en particulier par leurs points critiques. Soit f une fonction de Morse sur une variété compacte de dimension réelle n. Supposons que les points critiques de f sont isolés. Notons m_j, $0 \leq j \leq n$, le nombre de points critiques de f d'indice de Morse j et b_j le j-ième nombre de Betti de la variété. Les inégalités de Morse fortes assurent alors que pour $0 \leq q \leq n$

$$\sum_{j=0}^{q}(-1)^{q-j}b_j \leq \sum_{j=0}^{q}(-1)^{q-j}m_j, \tag{0.2.1}$$

aver égalité si $q = n$. D'après (0.2.1), on trouve les inégalités de Morse faibles :

$$b_j \leq m_j \qquad \text{pour} \quad 0 \leq j \leq n. \tag{0.2.2}$$

Dans son article fondateur [65], Witten donna une démonstration analytique des inégalités de Morse en étudiant le spectre de l'opérateur de Schrödinger $\Delta_t = \Delta + t^2|df|^2 + tV$, où $t > 0$ est un paramètre réel et V un opérateur d'ordre 0. Pour $t \to +\infty$, Witten montra que le spectre de Δ_t se rapproche, dans un certain sens, du spectre d'une somme d'oscillateurs harmonique attaché aux points critiques de f.

Dans [27], Demailly établit des inégalités de Morse asymptotiques analogues pour le complexe de Dolbeault associé aux grandes puissances tensorielles d'un fibré en droites hermitien holomorphe (L, h^L) sur une variété complexe (M, J). Les inégalités de Demailly donnent une borne asymptotique sur les sommes de Morse des nombres de Betti de $\bar{\partial}$ sur L^p en terme d'une intégrale de la courbure de Chern R^L de (L, h^L). Plus précisément, on définit $\dot{R}^L \in \mathrm{End}(T^{(1,0)}M)$ par la formule $g^{TM}(\dot{R}^L u, \overline{v}) = R^L(u, \overline{v})$ pour $u, v \in T^{(1,0)}M$, où g^{TM} est une métrique riemannienne J-invariante sur TM. On note $M(\leq q)$ l'ensemble des points où $\dot{R}^L$ est non-dégénérée et a au plus q valeurs propres négatives et on pose $n = \dim_{\mathbb{C}} M$. On a alors pour $0 \leq q \leq n$

$$\sum_{j=0}^{q}(-1)^{q-j} \dim H^j(M, L^p) \leq \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n + o(p^n), \tag{0.2.3}$$

avec égalité si $q = n$.

Ces inégalités ont eu de nombreuses applications. Par exemple, Demailly les utilisa dans [27] pour trouver une nouvelle caractérisation géométrique des variétés de Moishezon, améliorant ainsi la solution donnée par Siu [56, 57] de la conjecture de Grauert-Riemenschneider [36]. Une autre application notable des inégalités de Morse holomorphes est la démonstration par Siu [58, 29] du théorème de Matsusaka effectif. Récemment, Demailly a utilisé ces inégalités dans [30] pour effectuer un premier pas important dans la résolution d'une version généralisée de la conjecture de Green-Griffiths-Lang.

Pour démontrer ces inégalités, la remarque clef de Demailly fut que dans la formule du laplacien de Kodaira $\square_p$ associé à L^p, la métrique sur L joue formellement le rôle de la fonction de Morse dans l'article de Witten [65] et le paramètre p celui du paramètre t. La hessienne de la fonction de Morse devient alors la courbure de (L, h^L). La démonstration de Demailly est basée sur l'étude du comportement semi-classique quand $p \to +\infty$ de la fonction de comptage spectral de $\square_p$. Par la suite, Bismut redémontra dans [6] les inégalités de Morse holomorphes en adaptant sa démonstration utilisant le noyau de la chaleur des inégalités de Morse [5]. Le point clef est que l'on peut comparer le terme de gauche de (0.2.3) avec la trace alternée du noyau de la chaleur agissant sur les formes de degré inférieur à q, c'est-à-dire

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M,L^p) \leq \sum_{j=0}^{q}(-1)^{q-j}\operatorname{Tr}^{\Omega^{0,j}(M,L^p)}\Big[\exp\Big(-\frac{u}{p}\Box_p\Big)\Big], \tag{0.2.4}$$

avec égalité si $q = n$. Ensuite, Bismut obtint les inégalités de Morse holomorphes en montrant la convergence du noyau de la chaleur grâce à des techniques probabilistes. Demailly [28] et Bouche [20] donnèrent une approche analytique de ce dernier résultat. Dans [46], Ma et Marinescu donnèrent une nouvelle démonstration de cette convergence, remplaçant les arguments probabilistes de Bismut [6] par des arguments inspirés des techniques de localisation analytique de Bismut-Lebeau [14, Chap. 11].

Quand le fibré L est positif, (0.2.3) est une conséquence du théorème de Hirzebruch-Riemann-Roch et du théorème d'annulation de Kodaira, et se réduit à

$$\dim H^0(M,L^p) = \frac{p^n}{n!}\int_M \Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(p^n). \tag{0.2.5}$$

Dans ce cas, une estimation locale peut être obtenue en étudiant l'asymptotique du noyau de Bergman (le noyau de la projection orthogonale de $\mathscr{C}^\infty(M,L^p)$ sur $H^0(M,L^p)$) quand $p \to +\infty$. Nous renvoyons le lecteur au livre [46] et à ses références pour l'étude du noyau de Bergman.

Dans le cas équivariant, un groupe de Lie connexe compact G agit sur la variété M et son action se relève à L. Quand L est positif, Ma et Zhang [51] ont étudié le noyau de Bergman invariant, c'est-à-dire le noyau de la projection orthogonale de $\mathscr{C}^\infty(M,L^p)$ sur la partie G-invariante de $H^0(M,L^p)$. Soit μ l'application moment associée à l'action de G sur M (voir (0.2.7)). Ma et Zhang [51] établirent que le noyau de Bergman invariant se concentrait dans tout voisinage U de $\mu^{-1}(0)$, et que, près de $\mu^{-1}(0)$, on avait un développement asymptotique hors diagonale. Ils obtinrent aussi une décroissance rapide du noyau de Bergman invariant dans les directions normales à $\mu^{-1}(0)$, qui n'apparait pas dans le cas classique.

Dans ce chapitre, nous établissons une version G-invariant des inégalités de Morse holomorphes sous une certaine condition naturelle dans le contexte de Ma-Zhang [51], mais sans l'hypothèse de positivité de L.

Plu précisément, nous considérons une action d'un groupe de Lie connexe compact G sur une variété complexe M et deux fibrés G-équivariants L et E sur M, avec L de rang 1, et nous établissons des inégalités de Morse holomorphes similaires à (0.2.3) pour la partie G-invariante de la cohomologie de Dolbeault de $L^p \otimes E$ (voir les Théorèmes 0.2.3 et 0.2.5). Pour ce faire, nous définissons une « application moment » $\mu\colon M \to \mathrm{Lie}(G)$ par la formule de Kostant puis nous définissons la réduction de M sous une hypothèse naturelle sur $\mu^{-1}(0)$ (voir l'Hypothèse 0.2.1). Nos inégalités sont alors données en termes de la courbure du fibré induit par L sur cette réduction, et l'intégrale dans (0.2.3) portera sur des sous-ensembles de cette réduction.

Un nouvel élément dans notre cas, en comparaison avec le résultat de Demailly, est la localisation près de $\mu^{-1}(0)$. Nous utilisons une méthode par noyau de la chaleur inspirée de [6] (voir aussi [46, Sect. 1.6-1.7]), justifiée par le fait qu'un analogue de (0.2.4) est vérifié par la restriction du laplacien de Kodaira à l'espace des formes invariantes (*cf.* le Lemme 0.2.7). Nous montrons ici que le noyau de la chaleur se concentre dans tout voisinage U de $\mu^{-1}(0)$, et nous étudions son asymptotique près de $\mu^{-1}(0)$. Pour cette dernière partie, nous travaillons avec l'opérateur induit par le laplacien de Kodaira sur le quotient de

U. Cependant, comme nous devrons intégrer le noyau de la chaleur dans les directions normales à $\mu^{-1}(0)$, nous devons avoir un résultat de convergence plus précis que dans [46, Sect. 1.6]. En fait, nous montrons une décroissance rapide uniforme du noyau de la chaleur dans les directions normales, qui est analogue à la décroissance rencontrée dans [51, Thm. 0.2]. Notre approche est largement inspirée de [51].

Il est à noter que, dans la littérature, il existe un autre type d'inégalités de Morse holomorphes [66, 53, 67], qui relient la cohomologie de Dolbeault de l'ensemble des points fixes d'une variété kählerienne compacte M munie d'une action d'un groupe de Lie compact connexe à la cohomologie de Dolbeault de M elle-même.

Donnons maintenant plus de détails sur nos résultats. Soit (M, J) une variété complexe connexe, compacte et de dimension complexe n. Soit (L, h^L) un fibré en droites hermitien holomorphe sur M et (E, h^E) un fibré hermitien holomorphe sur M. Notons ∇^L et ∇^E les connexions de Chern de L et E, ainsi que $R^L = (\nabla^L)^2$ et $R^E = (\nabla^E)^2$ leur courbure. Soit ω la première forme de Chern de (L, h^L), c'est-à-dire la $(1,1)$-forme définie par

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.2.6}$$

On *ne suppose pas* que ω est une $(1,1)$-forme positive.

Soit G un groupe de Lie connexe compact, d'algèbre de Lie $\mathfrak{g}$ et de dimension réelle d. On suppose que G agit par biholomorphismes sur (M, J) et que cette action se relève en des actions holomorphes sur L et E. Nous supposons de plus que h^L et h^E sont préservées par G. Les formes R^L, R^E et ω sont alors invariantes.

Dans la suite, si F est une représentation de G, nous noterons F^G l'ensemble des éléments de F qui sont fixés par G. L'action infinitésimale de $K \in \mathfrak{g}$ sur F sera notée $\mathcal{L}_K^F$, ou simplement $\mathcal{L}_K$ quand cela n'entraîne pas de confusion.

Pour $K \in \mathfrak{g}$, soit K^M le champ de vecteurs sur M induit par K (voir (2.2.2)). Nous pouvons définir une application $\mu \colon M \to \mathfrak{g}^*$ par la formule de Kostant

$$\mu(K) = \frac{1}{2i\pi}\left(\nabla^L_{K^M} - \mathcal{L}_K\right). \tag{0.2.7}$$

Pour tout $K \in \mathfrak{g}$, on a alors $d\mu(K) = i_{K^M}\omega$. De plus, l'ensemble défini par

$$P = \mu^{-1}(0), \tag{0.2.8}$$

est stable par G.

Nous faisons l'hypothèse suivante :

Hypothèse 0.2.1. 0 *est une valeur régulière de* μ.

Sous l'Hypothèse 0.2.1, P est une sous-variété de M. De plus, par le Lemme 2.3.2, G agit localement librement sur P, de sorte que le quotient $M_G := P/G$ est une orbifold, que nous appelons la *réduction* de M. Pour la définition et les propriétés de base des orbifolds, nous renvoyons par exemple à [46, Sect. 5.4].

Notons TY le tangent aux orbites dans P. Comme G agit localement librement sur P, on sait que $TY = \mathrm{Vect}(K^M, K \in \mathfrak{g})$ et que c'est un fibré vectoriel sur P.

Nous montrerons l'analogue suivant de la réduction kählerienne classique (voir [37]).

Théorème 0.2.2. *La structure complexe J sur M induit structure complexe J_G sur M_G, pour laquelle les fibré orbifold L_G et E_G sur M_G induit par L et E sont holomorphes. De* *plus, la forme ω descend en une forme ω_G sur M_G et si R^{L_G} est la courbure de Chern de L_G pour la métrique h^{L_G} induite par h^L, alors*

$$\omega_G = \frac{\sqrt{-1}}{2\pi} R^{L_G}. \tag{0.2.9}$$

Enfin, π_* induit un isomorphisme

$$\ker \omega_G \simeq (\ker \omega)|_P. \tag{0.2.10}$$

Soit b^L la forme bilinéaire sur TM définie par

$$b^L(\cdot\,,\cdot) = \frac{\sqrt{-1}}{2\pi} R^L(\cdot\,, J\cdot) = \omega(\cdot\,, J\cdot). \tag{0.2.11}$$

Nous montrerons dans le Lemme 2.3.3 que b^L est non-dégénérée en restriction à $TY \times TY$ sur P. En particulier, la signature de $b^L|_{TY\times TY}$ est constante sur P. Nous noterons cette signature $(r, d-r)$, ce qui signifie que dans toute base orthogonale (pour b^L) de $TY|_P$, la matrice de b^L a r éléments diagonaux strictement négatifs et $d-r$ strictement positifs.

On définit $\dot{R}^{L_G} \in \mathrm{End}(T^{(1,0)}M_G)$ par $g(\dot{R}^{L_G}u, \overline{v}) = R^{L_G}(u, \overline{v})$ pour $u, v \in T^{(1,0)}M_G$, où g est une métrique riemannienne J_G-invariante sur le fibré tangent orbiflod TM_G. Notons $M_G(q)$ l'ensemble des points $x \in M_G$ en lesquels $\dot{R}^{L_G}$ est inversible et a exactement q valeur propres négatives, avec la convention que $M_G(q) = \emptyset$ si $q \notin \{0, \dots, n-d\}$. Posons $M_G(\leq q) = \cup_{i=0}^{q} M_G(i)$. Remarquons que $M_G(q)$ ne dépend pas de la métrique g.

Étant donné que G préserve toutes les structures que l'on s'est donné, il agit naturellement sur la cohomologie de Dolbeault $H^\bullet(M, L^p \otimes E)$. Le théorème suivant est le résultat principal de ce chapitre.

Théorème 0.2.3. *Supposons que G agit fidèlement sur M (c'est-à-dire l'identité est le seul élément de G agissant comme Id_M). Alors quand $p \to +\infty$, on a les inégalités de Morse fortes suivantes pour $q \in \{1, \dots, n\}$:*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \leq \mathrm{rk}(E) \frac{p^{n-d}}{(n-d)!} \int_{M_G(\leq q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}), \tag{0.2.12}$$

avec égalité si $q = n$.

En particulier, on obtient les inégalités de Morse faibles

$$\dim H^q(M, L^p \otimes E)^G \leq \mathrm{rk}(E) \frac{p^{n-d}}{(n-d)!} \int_{M_G(q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}). \tag{0.2.13}$$

Remarque 0.2.4. On suppose temporairement que G agit librement sur P, de sorte que M_G est une variété.

Si L est positif, alors ω est une forme kählerienne et μ est une vraie application moment. De plus, (M_G, ω_G) est la réduction kählerienne usuelle de M. D'après [62, Thm. 0.2], la quantification et la réduction commutent dans ce cas : pour p assez grand,

$$H^\bullet(M, L^p \otimes E)^G = H^\bullet(M_G, L_G^p \otimes E_G). \tag{0.2.14}$$

En particulier, comme dans le cas non équivariant, le Théorème 0.2.3 est une conséquence de (0.2.14) ainsi que des théorèmes de Hirzebruch-Riemann-Roch et d'annulation de Kodaira, appliqués cette fois à M_G.

Nous montrons ici que même si ω est dégénérée ou si G n'agit pas librement sur P, sous l'Hypothèse 0.2.1, on a une estimation similaire pour $\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G$.

Le Théorème 0.2.3 est en fait un cas particulier du Théorème 0.2.5 ci-dessous.

Définissons

$$G^0 = \{g \in G \,:\, \forall x \in M,\, gx = x\}, \tag{0.2.15}$$

qui est un sous-groupe distingué fini de G. Nous verrons en (2.6.23) que l'on a aussi $G^0 = \{g \in G \,:\, \forall x \in P,\, g \cdot x = x\}$.

Observons que $\dim(L^p_v \otimes E_v)^{G^0}$ ne dépend pas de $v \in M$. Nous noterons donc cette quantité simplement $\dim(L^p \otimes E)^{G^0}$.

Theorem 0.2.5. *quand $p \to +\infty$, on a les inégalités de Morse fortes suivantes pour $q \in \{1, \ldots, n\}$:*

$$\sum_{j=0}^{q}(-1)^{q-j} \dim H^j(M, L^p \otimes E)^G$$
$$\leq \dim(L^p \otimes E)^{G^0} \frac{p^{n-d}}{(n-d)!} \int_{M_G(\leq q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}), \tag{0.2.16}$$

avec égalité si $q = n$.

En particulier, on obtient les inégalités de Morse faibles

$$\dim H^q(M, L^p \otimes E)^G \leq \dim(L^p \otimes E)^{G^0} \frac{p^{n-d}}{(n-d)!} \int_{M_G(q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}). \tag{0.2.17}$$

Remarque 0.2.6. L'entier $\dim(L^p \otimes E)^{G^0}$ dépend de p. Cependant, comme G^0 est fini et agit par rotations sur L, il existe un $k \in \mathbb{N}$ (divisant le cardinal de G^0) tel que G^0 agisse trivialement sur L^k. On a donc $\dim(L^{kp} \otimes E)^{G^0} = \dim E^{G^0}$.

Expliquons à présent les principales étapes de la démonstration.

Soit g^{TM} une métrique J- et G-invariante sur TM. Soit dv_M le volume riemannien correspondant sur M, et soit ∇^{TM} la connexion de Levi-Civita de (TM, g^{TM}). Soit $\bar\partial^{L^p\otimes E}$ l'opérateur de Dolbeault agissant sur $\Omega^{0,\bullet}(M, L^p \otimes E)$. Soit $\bar\partial^{L^p\otimes E,*}$ son dual par rapport au produit L^2 induit par g^{TM}, h^L et h^E. Posons

$$D_p = \sqrt{2}\left(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\right) \tag{0.2.18}$$

et notons $e^{-uD_p^2}$ le noyau de la chaleur associé.

Notons P_G la projection orthogonale de $\Omega^{0,\bullet}(M, L^p \otimes E)$ sur $\Omega^{0,\bullet}(M, L^p \otimes E)^G$. Soit $(P_G e^{-\frac{u}{p}D_p^2} P_G)(v, v')$ le noyau de Schwartz de $P_G e^{-\frac{u}{p}D_p^2} P_G$ par rapport à $dv_M(v')$.

L'opérateur D_p^2 agit sur $\Omega^{0,\bullet}(M, L^p \otimes E)^G$ (c'est-à-dire commute avec P_G) et préserve la $\mathbb{Z}$-graduation. Notons $\mathrm{Tr}_q[P_G e^{-\frac{u}{p}D_p^2} P_G]$ la trace de $P_G e^{-\frac{u}{p}D_p^2} P_G$ agissant sur $\Omega^{0,q}(M, L^p \otimes E)$. On a alors l'analogue de (0.2.4) :

Théorème 0.2.7. *Pour tout $u > 0$, $p \in \mathbb{N}^*$ et $0 \leq q \leq n$, on a*

$$\sum_{j=0}^{q}(-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \leq \sum_{j=0}^{q}(-1)^{q-j} \mathrm{Tr}_j[P_G e^{-\frac{u}{p}D_p^2} P_G], \tag{0.2.19}$$

avec égalité pour $q = n$.

La suite consiste à estimer $P_G e^{-\frac{u}{p}D_p^2} P_G$ pour traiter le terme de droite de (0.2.19).

Soit U un petit voisinage ouvert et G-invariant de P, tel que G agisse localement librement sur son adhérence $\overline{U}$.

Tout d'abord, nous avons loin de P le théorème suivant. Le résultat correspondant de Ma-Zhang pour le noyau de Bergman est [51, Thm. 0.1].

Théorème 0.2.8. *Pour $u > 0$ et $k, \ell \in \mathbb{N}$ fixés, il existe $C > 0$ tel que pour tout $p \in \mathbb{N}^*$ et $v, v' \in M$ avec $v, v' \in M \setminus U$,*

$$\left| P_G e^{-\frac{u}{p} D_p^2} P_G(v, v') \right|_{\mathscr{C}^\ell} \leq C p^{-k}, \tag{0.2.20}$$

où $|\cdot|_{\mathscr{C}^\ell}$ est la norme $\mathscr{C}^\ell$induite par ∇^L, ∇^E, ∇^{TM}, h^L, h^E et g^{TM}.

Passons à l'estimation « près de P » du noyau de la chaleur. Pour expliquer plus simplement cette asymptotique, nous supposons ici que G agit librement sur P. Nous pouvons aussi supposer que G agit librement sur $\overline{U}$. Soit $B = U/G$. Alors M_G et B sont de vraies variétés. Nous expliquerons dans la Section 2.6.2 comment adapter la preuve des Théorèmes 0.2.3 et 0.2.5 dans le cas d'une action localement libre.

Par le Lemme 2.3.3, on a

$$TU = TY \oplus (TY)^{\perp_{b^L}}. \tag{0.2.21}$$

Par conséquent nous pouvons choisir pour fibré horizontal des fibrations $U \to B$ et $P \to M_G$ les fibrés

$$T^H U = (TY)^{\perp_{b^L}} \quad \text{et} \quad T^H P = T^H U\Big|_P \cap TP. \tag{0.2.22}$$

En effet, en utilisant (2.1.21) et le fait que $TY \subset TP$, on a

$$TP = TY \oplus T^H P. \tag{0.2.23}$$

Soit $g^{T^H P}$ une métrique G- et J-invariante sur $T^H P$. Soit g^{TY} une métrique G-invariante sur TY et soit g^{TY} la métrique G-invariante sur JTY induite par J et g^{TY}. D'après (2.3.19), on peut choisir la métrique g^{TM} de sorte qu'en restriction à P :

$$g^{TM}|_P = g^{TY}|_P \oplus g^{JTY}|_P \oplus g^{T^H P}. \tag{0.2.24}$$

Nous utiliserons cette hypothèse sur g^{TM} dans le reste de l'introduction ainsi que dans les Sections 2.5.1-2.6.2.

Supposons que U est assez petit pour être identifié à un ε-voisinage, $\varepsilon > 0$, de la section nulle du fibré normal N de P dans U via l'application exponentielle. Nous noterons les coordonnées correspondantes $v = (y, Z^\perp) \in U$ avec $y \in P$ and $Z^\perp \in N_y$. Remarquons que d'après (0.2.23) et (0.2.24), nous pouvons identifier N_y et JTY_y.

Soit $\mathbf{J} \in \mathrm{End}(TM|_P)$ tel que sur P

$$\omega = g^{TM}(\mathbf{J}\cdot, \cdot). \tag{0.2.25}$$

Nous noterons aussi $\mathbf{J}$ l'opérateur induit sur B.

Nous verrons avec (2.5.3) que le fibré normal N_G à M_G dans B peut être identifié avec le fibré $(JTY)_B$ induit sur B par JTY (voir la Section 2.2). En particulier, si $\pi(y) = x$, nous garderons la même notation pour un élément de N_y et l'élément lui correspondant dans $N_{G,x}$. Dans la Section 2.5.3, nous verrons que $\mathbf{J}$ laisse N_G stable et nous définirons $a_i^\perp \in \mathbb{R}^*$ tels que

$$\mathrm{Sp}(\mathbf{J}^2|_{N_G}) = -\frac{1}{4\pi^2}\{a_1^{\perp,2}, \dots, a_d^{\perp,2}\}. \tag{0.2.26}$$

Soit g^{TB} la métrique induite par g^{TM} et $T^H P$ sur TB. Soit g^{N_G} la métrique induite sur N_G et dv_{N_G} la forme volume correspondante. Pour $x \in M_G$, soit $\{e_i^\perp\}_{i=1}^d$ une base orthonormale de $N_{G,x}$ telles que $\mathbf{J}_x^2 e_i^\perp = -\frac{1}{4\pi^2} a_i^\perp(x) e_i^\perp$. Nous pouvons alors identifier $\mathbb{R}^d$ et $N_{G,x}$ via l'application

$$(Z_1^\perp, \dots, Z_d^\perp) \in \mathbb{R}^d \mapsto Z^\perp = \sum_{i=1}^d Z_i^\perp e_i^\perp. \tag{0.2.27}$$

Définissons l'opérateur $\mathscr{L}_x^\perp$ agissant sur $N_{G,x} \simeq \mathbb{R}^d$ par

$$\mathscr{L}_x^\perp = -\sum_{i=1}^d \Big((\nabla_{e_i^\perp})^2 - |a_i^\perp Z_i^\perp|^2 \Big) - \sum_{j=1}^d a_j^\perp, \tag{0.2.28}$$

où ∇_U désigne l'opérateur de différentiation ordinaire sur $\mathbb{R}^d$ dans la direction U. Notons $e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z'^\perp)$ le noyau de la chaleur de $\mathscr{L}_x^\perp$ par rapport à $dv_{N_{G,x}}(Z'^\perp)$.

Soit g^{TM_G} la métrique sur M_G induite par g^{TM} et T^HP et dv_{M_G} la forme volume correspondante. Notons $\langle\cdot\,,\cdot\rangle_G$ l'extension $\mathbb{C}$-bilinéaire de g^{TM_G} sur $T_{\mathbb{C}}M_G$. Nous identifions alors R^{L_G} à une matrice hermitienne $\dot{R}^{L_G} \in \mathrm{End}(T^{(1,0)}M_G)$ telle que pour $V, V' \in T^{(1,0)}M_G$,

$$R^{L_G}(V,V') = \langle \dot{R}^{L_G} V, \overline{V'}\rangle_G. \tag{0.2.29}$$

Soit $\{w_j\}$ une base othonormale locale de $T^{(1,0)}M_G$ de base duale $\{w^j\}$. Posons

$$\omega_{G,d} = -\sum_{i,j} R^{L_G}(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i}. \tag{0.2.30}$$

Soit h la fonction G-invariante sur M donnée par

$$h(x) = \sqrt{\mathrm{vol}(G.x)}, \tag{0.2.31}$$

et soit $\kappa \in \mathscr{C}^\infty(TB|_{M_G})$la fonction définie par $\kappa|_{M_G} = 1$ et pour $x \in M_G$, $Z \in T_xB$,

$$dv_B(x,Z) = \kappa(x,Z)dv_{T_{x_0}B}(Z) = \kappa(x,Z)dv_{M_G}(x)dv_{N_{G,x}}(Z). \tag{0.2.32}$$

Le résultat suivant est une version de [51, Thm. 2.21] pour le noyau de la chaleur dans notre situation.

Théorème 0.2.9. *Supposons que G agit librement sur P. Pour tout $u > 0$ et $m \in \mathbb{N}$ fixé, on a la convergence suivante, quand $p \to +\infty$ et pour $|Z^\perp| < \varepsilon$:*

$$\begin{aligned} h(y,Z^\perp)^2\big(P_G e^{-\frac{u}{p}D_p^2} P_G\big)\big((y,Z^\perp),(y,Z^\perp)\big) = \\ \frac{\kappa^{-1}(x,Z^\perp)}{(2\pi)^{n-d}} \frac{\det(\dot{R}_x^{L_G})e^{2u\omega_d(x)}}{\det\big(1-\exp(-2u\dot{R}_x^{L_G})\big)} e^{-u\mathscr{L}_x^\perp}(\sqrt{p}Z^\perp, \sqrt{p}Z^\perp) \otimes \mathrm{Id}_E\, p^{n-d/2} \\ + O\big(p^{n-d/2-1/2}(1+\sqrt{p}|Z^\perp|)^{-m}\big), \end{aligned} \tag{0.2.33}$$

où $x = \pi(y) \in M_G$ et le terme $O(\cdot)$ est uniforme. De plus, la convergence est pour la topologie $\mathscr{C}^\infty$ en la variable $y \in P$. Ici, nous utilisons la convention que si une valeur propre de $\dot{R}_{x_0}^{L_G}$ est nulle, alors sa contribution à $\frac{\det(\dot{R}_{x_0}^{L_G})}{\det\big(1-\exp(-u\dot{R}_{x_0}^{L_G})\big)}$ est $\frac{1}{2u}$.

Grâce aux Théorèmes 0.2.7, 0.2.8 et 0.2.9, on obtient le Théorème 0.2.3 dans le cas où G agit librement sur P, par intégration sur M.

0.3 Asymptotique des formes de torsion analytique holomorphe

La torsion analytique holomorphe a été définie dans [54] par Ray et Singer comme l'analogue complexe de la torsion réelle pour les fibrés vectoriels plats. Elle s'obtient comme déterminant régularisé du laplacien de Kodaira d'un fibré vectoriel holomorphe sur une variété complexe compacte. La torsion apparaît notamment dans l'étude du déterminant de la cohomologie de la fibre d'une fibration holomorphe menée par Bismut-Gillet-Soulé dans [12].

La torsion analytique a une extension dans le cas des familles : les formes de torsion analytique, définies d'abord par Bismut-Gillet-Soulé [11], puis par Bismut-Köhler [13] et Bismut [9], à divers degrés de généralité. La partie de degré zéro de ces formes n'est autre que la torsion de Ray-Singer le long de la fibre. Les formes de torsion analytique ont eu beaucoup d'applications, en particulier car elles ont été introduites, notamment par Gillet et Soulé, comme la contribution analytique au formalisme de l'image directe en géométrie d'Arakelov. En effet, la torsion apparaît dans le théorème de Riemann-Roch arithmétique [35] et les formes de torsion dans le théorème de Riemann-Roch-Grothendieck arithmétique en degré supérieur [34].

La torsion analytique a aussi d'autres extensions. Dans [39], Köhler et Roessler ont utilisé une version équivariante de la torsion pour démontrer un théorème de Riemann-Roch en K_0-théorie arithmétique équivariante. Récemment, Burgos Gil, Freixas i Montplet et Liţcanu ont étendu dans [23] les classes de torsion analytique holomorphe aux morphismes quelconques entre variétés algébriques lisses en donnant une définition axiomatique de ces classes. Ils ont aussi classifié les théories ainsi obtenues. Dans [22], les mêmes auteurs ont utilisé leur théorie des classes de torsion généralisées pour démontrer un théorème de Riemann-Roch-Grothendieck pour les morphismes projectifs généraux entre variétés arithmétiques régulières.

Dans [17], Bismut et Vasserot ont calculé l'asymptotique de la torsion analytique associée à des puissances croissantes d'un fibré en droites positif en utilisant la méthode du noyau de la chaleur de [6] (voir aussi [46, Sect. 5.5]). Dans [18], ils ont aussi étendu leur résultat dans le cas où les puissances du fibré en droites sont remplacées par les puissances symétrique d'un fibré positif (de rang quelconque) en utilisant une astuce due à Getzler [33]. Ces asymptotiques ont joué un rôle important dans un résultat d'amplitude arithmétique de Gillet et Soulé [35] (voir aussi [59, Chp VIII]).

Dans cette partie, nous donnons une version en familles, et au niveau des formes, des résultats de Bismut et Vasserot pour les formes de torsion analytique.

Nous considérons d'abord le cas des formes de torsion d'une fibration associées aux puissances d'un fibré en droites donné, positif le long des fibres. Ceci correspond à [17]. Nous utilisons ici une stratégie similaire à celle de cet article, mais des difficultés supplémentaires surviennent ici, dues aux formes différentielles horizontales qui apparaissent dans la superconnexion de Bismut (par rapport au laplacien de Kodaira) utilisée dans la définition des formes de torsion. En effet, les opérateurs que nous utilisons ici ont une partie nilpotente (la partie de degré non-nulle sur la base) qui doit être prise en compte, spécialement lors des estimations des résultantes ou des noyaux de la chaleur. De plus, pour donner un formule asymptotique, nous devons calculer explicitement des super-traces de termes contenant une exponentielle couplant formes horizontales et variables de Clifford verticales. Ces termes rendent le calcul bien plus compliqué que dans [17]. Notons aussi que dans tout nos résultats de convergence $\mathscr{C}^\infty$, nous devons prendre en compte les dérivées le long de la base.

Ensuite, nous considérons le cas des formes de torsion d'une fibration associées à l'image directe des puissances d'un fibré en droites sur une variété plus grosse. Nous devons pour cela faire une hypothèse de positivité partielle sur le fibré en droites. Notre résultat généralise [18] dans deux directions. Premièrement, nous travaillons en famille. Deuxièmement, il est facile de voir que le résultat de [18] s'applique en fait à l'image directe des puissances d'un fibré en droites sur une variété plus grosse quand cette variété est donnée par un G-

fibré principal avec G compact et connexe. Nous ne supposons pas que cela soit le cas ici, et par conséquent nous ne pouvons pas utiliser la même astuce que dans [18] pour réduire le problème à notre premier résultat. Dans le même esprit, quand la base est compacte kählerienne et que la fibration est kählerienne, nous montrerons comment associer notre premier résultat à [13] et [41] pour obtenir simplement l'asymptotique désirée modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$. Cependant cette méthode ne peut pas aboutir en général.

Dans le cas général, nous utilisons la même approche par le noyau de la chaleur que dans notre premier résultat. Cependant, en plus des difficultés indiquées ci-dessus, nous devons ici gérer le fait que la dimension du fibré sur lequel on travail croît vers l'infini. En particulier, nous ne pouvons espérer trouver un opérateur limite lors du changement d'échelle, pas plus que de coefficients limites pour le développement du noyau de la chaleur ; et dans toute nos démonstrations, nous devons avoir des estimations uniformes sur des espaces qui changent. Pour contourner ce problème, nous nous inspirons de [15, 16] en utilisant le formalisme des opérateurs de Toeplitz de [46]. L'idée est d'utiliser la norme d'opérateurs sur les matrices pour avoir des bornes uniformes sur les opérateurs de Toeplitz, et de remplacer la convergence vers des objets limites par l'approximation par des objets à coefficients opérateurs de Toeplitz.

Donnons à présent plus de détails sur nos résultats. Soit M et B deux variétés complexes. Soit $\pi\colon M \to B$ une fibration holomorphe dont la fibre X est compacte et de dimension complexe n. Dans ce chapitre, nous voulons être plus précis dans nos notations concernant les fibrés tangents. Aussi, nous noterons $T_{\mathbb{R}}X$ le tangent réel de la fibre et TX le tangent holomorphe, qui n'est autre que $T_{\mathbb{R}}X$ vu comme espace complexe grâce à la structure complexe $J^{T_{\mathbb{R}}X}$ de la fibre. Soit $T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes \mathbb{C}$ le fibré tangent complexifié et $T^{(1,0)}X, T^{(0,1)}X \subset T_{\mathbb{C}}X$ les espaces propres pour les valeurs propres $\pm\sqrt{-1}$ de la structure complexe. Rappelons que l'on a un isomorphisme canonique $TX \simeq T^{(1,0)}X$. Dans la suite, nous utiliserons des notations similaires pour les autres espaces tangents qui apparaissent.

Soit (π, ω) une structure de fibration hermitienne au sens de la Section 3.2.1, c'est-à-dire que ω est une $(1,1)$-forme lisse sur M qui induit une métrique hermitienne h^{TX} le long des fibres.

Soit (ξ, h^{ξ}) et (L, h^L) deux fibrés vectoriels hermitiens holomorphes sur M, avec L de rang 1. Notons R^L la courbure de Chern de (L, h^L). Nous faisons l'hypothèse de base suivante :

Hypothèse 0.3.1. *La* $(1,1)$*-forme* $\sqrt{-1}R^L$ *est positive le long des fibres, ce qui signifie que pour tout* $0 \neq U \in T^{(1,0)}X$*, on a*

$$R^L(U, \overline{U}) > 0. \tag{0.3.1}$$

Soit $\dot{R}^{X,L} \in \mathrm{End}(TX)$ la matrice hermitienne telle que pour tout $U, V \in T^{(1,0)}X$,

$$R^L(U, \overline{V}) = \langle \dot{R}^{X,L} U, V \rangle_{h^{TX}}. \tag{0.3.2}$$

Par l'Hypothèse 0.3.1, $\dot{R}^{X,L}$ est définie positive.

Nous supposons qu'il existe $p_0 \in \mathbb{N}$ tel que l'image directe $R^i\pi_*(\xi \otimes L^p)$ soit localement libre pour $p \geq p_0$ et $i \in \{1, \dots, n\}$, et s'annule pour $i > 0$.

Dans la suite, tous les résultats sont vrais pour $p \geq p_0$, et nous ne répéterons pas cette hypothèse.

Remarque 0.3.2. Si la base B est compacte, l'Hypothèse 0.3.1 implique que pour p assez grand l'image directe $R^i\pi_*(\xi \otimes L^p)$ est automatiquement localement libre et s'annule pour $i > 0$.

Munissons $\xi \otimes L^p$ de la métrique $h^{\xi\otimes L^p}$ induite par h^ξ et h^L. Nous pouvons alors définir (voir la Section 3.2) les formes de torsion analytique holomorphe $\mathscr{T}(\omega, h^{\xi\otimes L^p})$ associée à (π, ω) et $(\xi \otimes L^p, h^{\xi\otimes L^p})$.

Soit

$$\Theta^M = \frac{\sqrt{-1}}{2\pi} R^L \quad \text{et} \quad \Theta^X = \frac{\sqrt{-1}}{2\pi} R^L|_{T_{\mathbb{R}}X \times T_{\mathbb{R}}X} \tag{0.3.3}$$

les premières formes de Chern de (L, h^L) et $(L|_X, h^{L|_X})$ respectivement.

Pour α une forme sur B, nous notons $\alpha^{(k)}$ sa composante de degré k. Nous pouvons maintenant énoncer le premier résultat principal de cette partie, qui est une extension de [17] dans le cas des familles.

Théorème 0.3.3. *Soit $k \in \{0, \ldots, \dim B\}$. Alors la composante de degré $2k$ de la torsion $\mathscr{T}(\omega, h^{\xi\otimes L^p})$ associée à ω et $h^{\xi\otimes L^p}$ admet le développement asymptotique suivant quand $p \to +\infty$:*

$$p^{-k} \mathscr{T}(\omega, h^{\xi\otimes L^p})^{(2k)} = \frac{\mathrm{rk}(\xi)}{2} \left(\int_X \log \left[\det \left(\frac{p\dot{R}^{X,L}}{2\pi} \right) \right] e^{\Theta^M + (p-1)\Theta^X} \right)^{(2k)} + o(p^n), \tag{0.3.4}$$

pour la topologie de la convergence $\mathscr{C}^\infty$ sur les compacts de B.

Détaillons maintenant le second résultat de cette partie. Soit N, M et B trois variétés complexes. Soit $\pi_1 \colon N \to M$ et $\pi_2 \colon M \to B$ deux fibration holomorphes, dont nous supposons les fibres respectives Y et X compactes. On a alors une troisième fibration holomorphe, $\pi_3 := \pi_2 \circ \pi_1 \colon N \to B$, dont la fibre est compacte et notée Z. Notons n_X (resp. n_Y, n_Z) la dimension complexe de X (resp. Y, Z). Il est à noter que $\pi_1|_Z \colon Z \to X$ est une fibration holomorphe de fibre Y. La situation est résumée dans le diagramme suivant :

$$\begin{tikzcd}
Y \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"] & N \arrow[d, "\pi_1"] \arrow[dr, "\pi_3"] & \\
& X \arrow[r] & M \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Donnons nous (π_2, ω^M) une structure de fibration hermitienne (voir la Section 3.2.1). Notons $T^H_B M = TX^{\perp, \omega^M}$ l'espace horizontal correspondant.

Soit (ξ, h^ξ) un fibré hermitien holomorphe sur M, et soit (η, h^η) et (L, h^L) deux fibrés vectoriels hermitiens holomorphes sur N, avec L de rang 1. Notons ∇^L la connexion de Chern de (L, h^L) et R^L la courbure correspondante.

Comme ci-dessus, nous faisons une hypothèse de positivité sur L :

Hypothèse 0.3.4. *La (1,1)-forme $\sqrt{-1}R^L$ est positive le long des fibres de π_3, ce qui signifie que, pour tout $0 \neq U \in TZ$,*

$$R^L(U, \overline{U}) > 0. \tag{0.3.5}$$

En particulier, $\frac{\sqrt{-1}}{2\pi} R^L$ nous permet de définir des métriques $g^{T_{\mathbb{R}}Z}$ et $g^{T_{\mathbb{R}}Y}$ sur $T_{\mathbb{R}}Z$ et $T_{\mathbb{R}}Y$ (voir (3.4.1)).

Supposons qu'il existe $p_0 \in \mathbb{N}$ tel que, pour tout $p \geq p_0$, l'image directe $R^\bullet \pi_{1,*}(\eta \otimes L^p)$ soit localement libre et $R^i \pi_{1,*}(\eta \otimes L^p) = 0$ si $i > 0$. Pour $p \geq p_0$,

$$F_p := H^0(Y, (\eta \otimes L^p)|_Y) \tag{0.3.6}$$

définit donc un fibré holomorphe $\mathbb{Z}$-gradué sur M (voir la Section 3.2.3), qui est muni d'une métrique L^2, notée h^{F_p}, induite par h^η, h^L et $g^{T_\mathbb{R}Y}$.

Pour $p \geq p_0$, nous supposons aussi que les images directes $R^\bullet\pi_{2,*}(\xi\otimes F_p)$ et $R^\bullet\pi_{3,*}(\pi_1^*\xi\otimes\eta\otimes L^p)$ sont localement libres. On a alors pour $i \geq 0$

$$R^i\pi_{2,*}(\xi\otimes F_p) \simeq R^i\pi_{3,*}(\pi_1^*\xi\otimes\eta\otimes L^p). \tag{0.3.7}$$

Remarque 0.3.5. Si la base B est compacte, l'Hypothèse 0.3.4 implique l'existence d'un p_0 tel que pour $p \geq p_0$ les conditions ci-dessus soient satisfaites : $R^\bullet\pi_{1,*}(\xi\otimes L^p)$, $R^\bullet\pi_{2,*}(\xi\otimes F_p)$ et $R^\bullet\pi_{3,*}(\pi_1^*\xi\otimes\eta\otimes L^p)$ sont localement libres et s'annulent en degré non nul. En particulier,

$$H^\bullet(X,(\xi\otimes F_p)|_X) = H^0(X,(\xi\otimes F_p)|_X) \simeq H^0(Z,(\pi_1^*\xi\otimes\eta\otimes L^p)|_Z). \tag{0.3.8}$$

Ici encore, tous les résultats sont vrais pour $p \geq p_0$, et nous ne répéterons pas cette hypothèse.

Munissons $\xi\otimes F_p$ de la métrique $h^{\xi\otimes F_p}$ induite par h^ξ et h^{F_p}. Nous pouvons alors construire, comme dans la Section 3.2, les formes de torsion $\mathscr{T}(\omega^M, h^{\xi\otimes F_p})$ associées à ω^M et $(\xi\otimes F_p, h^{\xi\otimes F_p})$.

Soit

$$T_B^H N = (TZ)^\perp, \quad T_M^H N = (TY)^\perp, \tag{0.3.9}$$

où l'orthogonal est pris par rapport à R^L. Alors

$$T_X^H Z := T_M^H N \cap TZ \tag{0.3.10}$$

est l'orthogonal de TY dans TZ. De plus,

$$T_B^H N \simeq \pi_3^* TB, \quad T_M^H N \simeq \pi_1^* TM \quad \text{et} \quad T_X^H Z \simeq \pi_1^* TX. \tag{0.3.11}$$

Remarque 0.3.6. Les fibrations $(\pi_1, g^{T_\mathbb{R}Y}, T_M^H N)$ et $(\pi_1|_Z, g^{T_\mathbb{R}Y}, T_X^H Z)$ sont des fibrations kähleriennes au sens de la Section 3.2.5.

Soit $\dot{R}^{X,L} \in \pi_3^*\mathrm{End}(TX)$ la matrice hermitienne telle que pour $U,V\in TX$, si $U^H, V^H \in T_X^H Z$ désignent les relevés horizontaux de U,V, alors

$$R^L(U^H, \overline{V}^H) = \langle \dot{R}^{X,L}U, V\rangle_{h^{TX}}. \tag{0.3.12}$$

Par l'Hypothèse 0.3.4, $\dot{R}^{X,L}$ est alors définie positive.

Soit

$$\Theta^N = \frac{\sqrt{-1}}{2\pi}R^L \quad \text{et} \quad \Theta^Z = \frac{\sqrt{-1}}{2\pi}R^L|_{T_\mathbb{R}Z\times T_\mathbb{R}Z}. \tag{0.3.13}$$

Nous pouvons maintenant énoncer le second résultat principal de cette partie, qui est une extension du Théorème 0.3.3 et une version en famille de [18] (voir l'introduction de la Section 3.4).

Théorème 0.3.7. *Soit $k \in \{0,\dots,\dim B\}$. Alors la composante de degré $2k$ de la torsion $\mathscr{T}(\omega, h^{\xi\otimes F_p})$ associée à ω^M et $h^{\xi\otimes F_p}$ admet le développement asymptotique suivant quand $p\to+\infty$:*

$$p^{-k}\mathscr{T}(\omega^M, h^{\xi\otimes F_p})^{(2k)} = \frac{\mathrm{rk}(\xi)\mathrm{rk}(\eta)}{2}\left(\int_Z \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right] e^{\Theta^N+(p-1)\Theta^Z}\right)^{(2k)} + o(p^{n_Z}), \tag{0.3.14}$$

pour la topologie de la convergence $\mathscr{C}^\infty$ sur les compacts de B.

Comme nous le verrons dans la Section 3.4, nous allons utiliser le formalisme des opérateurs de Toeplitz pour démontrer ce théorème. Rappelons la définition d'un opérateur de Toeplitz donnée dans [46, Def. 7.2.1].

Soit $b \in B$. Pour $x \in X_b := \pi_2^{-1}(b)$, soit $P_{p,x}$ la projection orthogonale

$$P_{p,x} \colon L^2(Y_x, \eta \otimes L^p) \to H^0(Y_x, \eta \otimes L^p), \tag{0.3.15}$$

où $Y_x := \pi_1^{-1}(x)$.

Définition 0.3.8. Un *opérateur de Toeplitz* sur Y_x est une famille d'opérateur $T_p \in \mathrm{End}(L^2(Y_x, \eta \otimes L^p))$ vérifiant les deux propriétés suivantes :

(i) pour tout $p \in \mathbb{N}$, on a
$$T_p = P_{p,x} T_p P_{p,x}; \tag{0.3.16}$$

(ii) il existe une suite $f_r \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta))$ telle que, pour tout $k \in \mathbb{N}$, il existe $C_k > 0$ telle que
$$\left\| T_p - \sum_{r=0}^{k} p^{-r} P_{p,x} f_r P_{p,x} \right\|_\infty \leq C_k p^{-k-1}. \tag{0.3.17}$$

Au cours de la démonstration du Théorème 0.3.7, un résultat intermédiaire important sera que le noyau de la chaleur associé à la superconnexion de Bismut est asymptote à un opérateur de Toeplitz. Précisons maintenant ce résultat. Soit $B_{u,p}$ la superconnexion de Bismut associée à ω^M et $(\xi \otimes F_p, h^{\xi \otimes F_p})$ (voir la Section 3.2.2). Par le Théorème 3.2.8, $B_{u,p}$ est un opérateur elliptique du second ordre le long de la fibre. Soit $\exp(-B^2_{p,u/p})$ le noyau de la chaleur correspondant. Pour $b \in B$, soit $\exp(-B^2_{p,u/p})(x, x')$ le noyau de Schwartz lisse de $\exp(-B^2_{p,u/p})$ par rapport à $dv_{X_b}(x')$. On a alors

$$\exp(-B^2_{p,u/p})(x, x) \in \mathrm{End}\left(\Lambda^\bullet_b(T^*_\mathbb{R} B) \otimes \left(\Lambda^{0,\bullet}(T^* X_b) \otimes \xi \otimes F_p\right)\right). \tag{0.3.18}$$

Pour $a > 0$, soit ψ_a l'automorphisme de $\Lambda(T^*_\mathbb{R} B)$ tel que pour $\alpha \in \Lambda^q(T^*_\mathbb{R} B)$

$$\psi_a \alpha = a^q \alpha. \tag{0.3.19}$$

Soit Ω_u la forme définie dans (3.4.172). Nous montrons alors :

Théorème 0.3.9. *Soit $k \in \mathbb{N}$. Pour $p \to +\infty$, on a l'asymptotique suivant pour la norme d'opérateur sur* $\mathrm{End}\left(\Lambda^\bullet_b(T^*_\mathbb{R} B) \otimes (\Lambda^{0,\bullet}(T^* X_b) \otimes \xi \otimes F_p)\right)$ *et la norme d'opérateur pour les dérivées d'ordres inférieurs à k en les variables $(b, x) \in M$, uniformément pour u dans un compact de $\mathbb{R}^*_+$,*

$$\begin{aligned} &\psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})(x, x) \\ &\qquad = \frac{p^{n_X}}{(2\pi)^{n_X}} P_{p,x} e^{-\Omega_{u,(x,\cdot)}} \frac{\det(\dot{R}^{X,L}_{(x,\cdot)})}{\det\left(1 - \exp(-u \dot{R}^{X,L}_{(x,\cdot)})\right)} \otimes \mathrm{Id}_{\xi_x} P_{p,x} + o(p^{n_X}). \end{aligned} \tag{0.3.20}$$

Ici, le point désigne la variable dans Y_x.

Remarque 0.3.10. La preuve du Théorème 0.3.9 que nous donnons ici n'utilise en réalité pas le fait que L est positif le long de la fibre Z, mai seulement le long de la fibre Y.

Chapter 1

The first terms in the expansion of the Bergman kernel in higher degrees

1.1 Introduction

The Bergman kernel of a Kähler manifold endowed with a positive line bundle L is the smooth kernel of the orthogonal projection on the kernel of the Kodaira Laplacian $\square^L = \bar{\partial}^L \bar{\partial}^{L,*} + \bar{\partial}^{L,*} \bar{\partial}^L$. The existence of a diagonal asymptotic expansion of the Bergman kernel associated with the p^{th} tensor power of L when $p \to +\infty$ and the form of the leading term were proved in [61], [24] and [69]. Moreover, the coefficients in this expansion encode geometric information about the underlying manifold, and therefore they have been studied closely: the second and third terms were computed by Lu [40], X. Wang [64], L. Wang [63] and by Ma-Marinescu [49] in different degrees of generality (see also the recent paper [68]). This asymptotics plays an important role in various problems of Kähler geometry; see for instance [31] or [32]. We refer the reader to the book [46] for a comprehensive study of the Bergman kernel and its applications. See also the survey [43].

In fact, Dai-Liu-Ma established the asymptotic development of the Bergman kernel in the symplectic case in [26], using the heat kernel (cf. also Ma-Marinescu [45]). Recently, this asymptotics in the symplectic case found an application in the study of variation of Hodge structures of vector bundles by Charbonneau and Stern in [25]. In their setting, the Bergman kernel is the kernel of a Kodaira-like Laplacian on a twisted bundle $L \otimes E$, where E is a (not necessarily holomorphic) complex vector bundle. Because of that, the Bergman kernel is no longer supported in degree 0 (unlike it did in the Kähler case), and the asymptotic development of its restriction to the $(0, 2j)$-forms is related to the degree of 'non-holomorphicity'of E.

In this chapter, we will show that the leading term in the asymptotics of the restriction to the $(0, 2j)$-forms of the Bergman kernel is of order $p^{\dim X - 2j}$ and we will compute it. That will lead to a local version of [25, (1.3)], which is the main technical results of their paper; see Remark 1.1.6. After that, we will also compute the second term in this asymptotics, as well as the third term in the case where the first two vanish.

We now give more detail about our results. Let (X, ω, J) be a compact Kähler manifold of complex dimension n. Let (L, h^L) be a holomorphic Hermitian line bundle on X, and (E, h^E) a Hermitian complex vector bundle. We endow (L, h^L) with its Chern (i.e., holomorphic and Hermitian) connection ∇^L, and (E, h^E) with a Hermitian connection

∇^E, whose curvatures are respectively $R^L = (\nabla^L)^2$ and $R^E = (\nabla^E)^2$.

Except in the beginning of Section 1.2.1, we will always assume that (L, h^L, ∇^L) satisfies the *pre-quantization condition*:

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1.1}$$

Let $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω and J. It induces a metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$, see Section 1.2.1.

Let $L^p = L^{\otimes p}$ be the p^{th} tensor power of L. Let $\Omega^{0,\bullet}(X, L^p \otimes E) = \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes E)$ and $\bar\partial^{L^p\otimes E} : \Omega^{0,\bullet}(X, L^p \otimes E) \to \Omega^{0,\bullet+1}(X, L^p \otimes E)$ be the Dolbeault operator induced by the $(0,1)$-part of ∇^E (cf. (1.2.3)). Let $\bar\partial^{L^p\otimes E,*}$ be its dual with respect to the L^2-product. We set (see (1.2.6)):

$$D_p = \sqrt{2}\Big(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\Big), \tag{1.1.2}$$

which exchanges odd and even forms.

Definition 1.1.1. Let

$$P_p \colon \Omega^{0,\bullet}(X, L^p \otimes E) \to \ker(D_p) \tag{1.1.3}$$

be the orthogonal projection onto the kernel $\ker(D_p)$ of D_p. The operator P_p is called the *Bergman projection*. It has a smooth kernel with respect to $dv_X(y)$, denoted by $P_p(x,y)$, which is called the *Bergman kernel*.

Remark 1.1.2. If E is holomorphic, then by Hodge theory and the Kodaira vanishing theorem (see respectively [46, Theorem 1.4.1] and [46, Theorem 1.5.6]), we know that, for p large enough, P_p is the orthogonal projection $\mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$. Here, by [44, Theorem 1.1], we just know that $\ker(D_p|_{\Omega^{0,\mathrm{odd}}(X,L^p\otimes E)}) = 0$ for p large, so that $P_p \colon \Omega^{0,\mathrm{even}}(X, L^p \otimes E) \to \ker(D_p)$. In particular, $P_p(x,x) \in \mathscr{C}^\infty\left(X, \mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E)\right)$.

By Theorem 1.2.3, D_p is a Dirac operator, which enables us to apply the following result:

Theorem 1.1.3 (Dai-Liu-Ma, [26, Thm. 1.1])**.** *There exist $\boldsymbol{b}_r \in \mathscr{C}^\infty(X, \mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E))$ such that for any $k \in \mathbb{N}$ and for $p \to +\infty$:*

$$p^{-n} P_p(x,x) = \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} + O(p^{-k-1}), \tag{1.1.4}$$

that is for every $k, l \in \mathbb{N}$, there exists a constant $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,

$$\left| p^{-n} P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} \right|_{\mathscr{C}^l(X)} \leq C_{k,l}\, p^{-k-1}. \tag{1.1.5}$$

Here $|\cdot|_{\mathscr{C}^l(X)}$ is the $\mathscr{C}^l$-norm for the variable $x \in X$.

To simplify the formulas, we will denote by

$$\mathscr{R} = (R^E)^{0,2} \in \Omega^{0,2}(X, \mathrm{End}(E)) \tag{1.1.6}$$

the $(0,2)$-part of R^E (which is zero if E is holomorphic). For $j \in [\![1, n]\!]$, let

$$I_j \colon \Lambda^{0,\bullet}(T^*X) \otimes E \to \Lambda^{0,j}(T^*X) \otimes E \tag{1.1.7}$$

be the natural orthogonal projection. The first main result in this chapter is

Theorem 1.1.4. *For any $k \in \mathbb{N}$, $k \geq 2j$, we have when $p \to +\infty$:*

$$p^{-n} I_{2j} P_p(x,x) I_{2j} = \sum_{r=2j}^{k} I_{2j} \boldsymbol{b}_r(x) I_{2j}\, p^{-r} + O(p^{-k-1}), \tag{1.1.8}$$

and moreover,

$$I_{2j} \boldsymbol{b}_{2j}(x) I_{2j} = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} I_{2j} \left(\mathscr{R}_x^j\right) \left(\mathscr{R}_x^j\right)^* I_{2j}, \tag{1.1.9}$$

where $\left(\mathscr{R}_x^j\right)^*$ is the dual of $\mathscr{R}_x^j$ acting on $(\Lambda^{0,\bullet}(T^*X) \otimes E)_x$.

Theorem 1.1.4 leads immediately to

Corollary 1.1.5. *Uniformly in $x \in X$, when $p \to +\infty$, we have*

$$\operatorname{Tr}\big((I_{2j} P_p I_{2j})(x,x) \big) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \left\| \mathscr{R}_x^j \right\|^2 p^{n-2j} + O(p^{n-2j-1}). \tag{1.1.10}$$

Remark 1.1.6. By integrating (1.1.10) over X, we get

$$\operatorname{Tr}(I_{2j} P_p I_{2j}) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \left\| \mathscr{R}^j \right\|_{L^2}^2 p^{n-2j} + O(p^{n-2j-1}), \tag{1.1.11}$$

which is the main technical result of Charbonneau and Stern [25, (1.3)]; thus Corollary 1.1.5 can be viewed as a local version of [25, (1.3)]. The constant in (1.1.11) differs from the one in [25] because our conventions are not the same as theirs (e.g., they choose $\omega = \sqrt{-1} R^L$, etc...).

Let $R^E_\Lambda := -\sqrt{-1} \sum_i R^E(w_i, \overline{w}_i)$ for $(\overline{w}_1, \dots, \overline{w}_n)$ an orthonormal frame of $T^{(0,1)}X$. Let R^{TX} be the curvature of the Levi-Civita connection ∇^{TX} of (X, g^{TX}), and for $(e_1, \dots, e_{2n})$ an orthonormal frame of TX, let $r^X = -\sum_{i,j} \langle R^{TX}(e_i, e_j) e_i, e_j \rangle$ be the scalar curvature of X.

For $j, k \in \mathbb{N}$ and $j \geq k$, we also define $\mathrm{C}_j(k)$ by

$$\mathrm{C}_j(k) := \frac{1}{(4\pi)^j} \frac{1}{2^k k!} \frac{1}{\prod_{s=k+1}^{j} (2s+1)}, \tag{1.1.12}$$

with the convention that $\prod_{s \in \emptyset} = 1$.

Let $\nabla^{\Lambda^{0,\bullet}}$ be the connection on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX}. Let $\nabla^{\Lambda^{0,\bullet} \otimes E}$ be the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by ∇^E and $\nabla^{\Lambda^{0,\bullet}}$, and let $\Delta^{\Lambda^{0,\bullet} \otimes E}$ be the associated Laplacian. For precise definitions, see Section 1.2.1.

For every operators A acting on a Hermitian space, we will define $\mathrm{Pos}[A]$ (resp. $\mathrm{Sym}[A]$) the positive (non necessarily definite) operator (resp. the symmetric operator) associated to A:

$$\mathrm{Pos}[A] = AA^* \quad \text{and} \qquad \mathrm{Sym}[A] = A + A^*. \tag{1.1.13}$$

Finally, to simplify the notation, we will define $\mathcal{T}_0(j)$, $\mathcal{T}_1(j)$, $\mathcal{T}_2(j)$ and $\mathcal{T}_3(j)$ as follows:

- $\mathcal{T}_0(0) = 0$, and for $j \geq 1$,

$$\mathcal{T}_0(j) = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{n} \sum_{k=0}^{j-1} I_{2j} \Big(\mathrm{C}_j(j) - \mathrm{C}_j(k) \Big) \mathscr{R}_x^{j-k-1} (\nabla^{\Lambda^{0,\bullet} \otimes E}_{\overline{w}_i} \mathscr{R}.)(x) \mathscr{R}_x^k I_0. \tag{1.1.14}$$

- $\mathcal{T}_1(0) = \mathcal{T}_1(1) = 0$, and for $j \geq 2$,

$$\mathcal{T}_1(j) = \frac{I_{2j}}{2\pi}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\Big\{\Big(\mathrm{C}_j(j) - \mathrm{C}_j(q+1)\Big)\mathscr{R}_x^{j-(q+2)}(\nabla_{\overline{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{q-m}(\nabla_{w_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^m$$
$$+ \mathrm{C}_j\left(m\right)\left[\prod_{s=q+2}^{j}\Big(1+\frac{1}{2s}\Big)-1\right]\mathscr{R}_x^{j-(q+2)}(\nabla_{w_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{q-m}(\nabla_{\overline{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^m\Big\}\, I_0, \tag{1.1.15}$$

- $\mathcal{T}_2(0) = 0$, and for $j \geq 1$,

$$\mathcal{T}_2(j) = \frac{1}{4\pi}I_{2j}\sum_{k=0}^{j-1}\Big\{\Big(\mathrm{C}_j(k) - \mathrm{C}_j(j)\Big)\mathscr{R}_x^{j-(k+1)}(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^k\Big\}\, I_0, \tag{1.1.16}$$

- for $j \geq 0$,

$$\mathcal{T}_3(j) = I_{2j}\sum_{k=0}^{j}\mathscr{R}_x^{j-k}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r_x^X - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x}\right]\mathscr{R}_x^k I_0. \tag{1.1.17}$$

The second goal of this chapter is to compute the second term in the expansion (1.1.8).

Theorem 1.1.7. *We can decompose $I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j}$ as the sum of four terms:*

$$I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j} = \mathrm{Pos}[\mathcal{T}_0(j)] + \mathrm{C}_j(j)\mathrm{Sym}\left[(\mathcal{T}_1(j)+\mathcal{T}_2(j)+\mathcal{T}_3(j))(\mathscr{R}_x^j)^* I_{2j}\right]. \tag{1.1.18}$$

For instance, for $j=1$, using the fact that $(R^E_\Lambda)^* = R^E_\Lambda$, we find

$$128\pi^3 I_2\boldsymbol{b}_3(x)I_2 = \frac{1}{9}\mathrm{Pos}\left[I_2\sum_{i=0}^{n}(\nabla_{\overline{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)I_0\right] - \frac{1}{6}\mathrm{Sym}\left[I_2(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^* I_2\right]$$
$$-\frac{\sqrt{-1}}{6}I_2(R^E_\Lambda\mathscr{R}_x\mathscr{R}_x^* + \mathscr{R}_x\mathscr{R}_x^* R^E_\Lambda)I_2 - \frac{2\sqrt{-1}}{3}I_2\mathscr{R}_x R^E_\Lambda\mathscr{R}_x^* I_2 - \frac{r_x^X}{4}I_2\mathscr{R}_x\mathscr{R}_x^* I_2. \tag{1.1.19}$$

The last goal of this chapter is to compute the third term in the expansion (1.1.8), assuming that the first two vanish.

Theorem 1.1.8. *Let $j \in [\![1,n]\!]$. If*

$$I_{2j}\boldsymbol{b}_{2j}(x)I_{2j} = I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j} = 0, \tag{1.1.20}$$

then $\mathcal{T}_3$ equals

$$\mathcal{T}_3'(j) := -\sqrt{-1}I_{2j}\sum_{k=0}^{j}\frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\mathscr{R}_x^{j-k}R^E_{\Lambda,x}\mathscr{R}_x^k I_0, \tag{1.1.21}$$

and

$$I_{2j}\boldsymbol{b}_{2j+2}(x)I_{2j} = \mathrm{Pos}[\mathcal{T}_1(j)+\mathcal{T}_2(j)+\mathcal{T}_3'(j)]. \tag{1.1.22}$$

Theorems 1.1.4, 1.1.7 and 1.1.8 yields to the following result.

Corollary 1.1.9. *We have:*

$$I_{2j}P_p(x,x)I_{2j} = O(p^{n-2j-3}) \iff \begin{cases}\mathscr{R}_x^j = 0,\\ \mathcal{T}_0(j) = 0,\\ \mathcal{T}_1(j)+\mathcal{T}_2(j)+\mathcal{T}_3'(j) = 0.\end{cases} \tag{1.1.23}$$

This chapter is organized as follows. In Section 1.2 we compute the square of D_p and use a local trivialization to rescale it, and then give the Taylor expansion of the rescaled operator. In Section 1.3, we use this expansion to give a formula for the coefficients $\boldsymbol{b}_r$ appearing in (1.1.4), which will lead to a proof of Theorem 1.1.4. In Section 1.4, we prove Theorem 1.1.7 using again the formula for $\boldsymbol{b}_r$. Finally, in Section 1.5, we prove Theorem 1.1.8 using the technics and results of the preceding sections.

In this whole chapter, when an index variable appears twice in a single term, it means that we are summing over all its possible values.

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1.2 Rescaling D_p^2 and Taylor expansion

In this section, we follow the method of [46, Chapter 4], that enables to prove the existence of $\boldsymbol{b}_r$ in (1.1.4) in the case of a holomorphic vector bundle E, and that still applies here (as pointed out in [46, Section 8.1.1]). Then, in section 3 and 4, we will use this approach to understand $I_{2j}\boldsymbol{b}_r I_{2j}$ and prove Theorems 1.1.4 and 1.1.7.

In Section 1.2.1, we will first prove Theorem 1.2.3, and then give a formula for the square of D_p, which will be the starting point of our approach.

In Section 1.2.2, we will rescale the operator D_p^2 to get an operator $\mathscr{L}_t$, and then give the Taylor expansion of the rescaled operator.

In Section 1.2.3, we will study more precisely the limit operator $\mathscr{L}_0$.

1.2.1 The square of D_p

For further details on the material of this subsection, the lector can read [46]. First of all let us give some notations.

The Riemannian volume form of (X, g^{TX}) is given by $dv_X = \omega^n/n!$. We will denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes \mathbb{C}$ induced by g^{TX}.

For the rest of this Section 1.2.1, we will fix $(w_1, \dots, w_n)$ a local orthonormal frame of $T^{(1,0)}X$ with dual frame $(w^1, \dots, w^n)$. Then $(\overline{w}_1, \dots, \overline{w}_n)$ is a local orthonormal frame of $T^{(0,1)}X$ whose dual frame is denoted by $(\overline{w}^1, \dots, \overline{w}^n)$, and the vectors

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \text{ and } e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j) \tag{1.2.1}$$

form a local orthonormal frame of TX.

We choose the Hermitian metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$ such that $\{\overline{w}^{j_1} \wedge \dots \wedge \overline{w}^{j_k} / 1 \leq j_1 < \dots < j_k \leq n\}$ is an orthonormal frame of $\Lambda^{0,\bullet}(T^*X)$.

For any Hermitian bundle (F, h^F) over X, let $\mathscr{C}^\infty(X, F)$ be the space of smooth sections of F. It is endowed with the L^2-Hermitian metric:

$$\langle s_1, s_2\rangle = \int_X \langle s_1(x), s_2(x)\rangle_{h^F} dv_x(x). \tag{1.2.2}$$

The corresponding norm will be denoted by $\|\cdot\|_{L^2}$, and the completion of $\mathscr{C}^\infty(X, F)$ with respect to this norm by $L^2(X, F)$.

Let $\bar{\partial}^E$ be the Dolbeault operator of E: it is the $(0,1)$-part of the connection ∇^E

$$\bar{\partial}^E := \left(\nabla^E\right)^{0,1} : \mathscr{C}^\infty(X,E) \to \mathscr{C}^\infty(X, T^{*(0,1)}X \otimes E). \tag{1.2.3}$$

We extend it to get an operator

$$\bar{\partial}^E \colon \Omega^{0,\bullet}(X,E) \to \Omega^{0,\bullet+1}(X,E) \tag{1.2.4}$$

by the Leibniz formula: for $s \in \mathscr{C}^\infty(X,E)$ and $\alpha \in \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X))$ homogeneous,

$$\bar{\partial}^E(\alpha \otimes s) = (\bar{\partial}\alpha) \otimes s + (-1)^{\deg \alpha} \alpha \otimes \bar{\partial}^E s. \tag{1.2.5}$$

We can now define the operator

$$D^E = \sqrt{2}\left(\bar{\partial}^E + \bar{\partial}^{E,*}\right) : \Omega^{0,\bullet}(X,E) \to \Omega^{0,\bullet}(X,E), \tag{1.2.6}$$

where the dual is taken with respect to the L^2- norm associated with the Hermitian metrics $h^{\Lambda^{0,\bullet}}$ and h^E.

Let $\nabla^{\Lambda(T^*X)}$ be the connection on $\Lambda(T^*X)$ induced by the Levi-Civita connection ∇^{TX} of X. Since X is Kähler, ∇^{TX} preserves $T^{(0,1)}X$ and $T^{(1,0)}X$. Thus, it induces a connection $\nabla^{T^{*(0,1)}X}$ on $T^{*(0,1)}X$, and then a Hermitian connection $\nabla^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X)$. We then have that for any $\alpha \in \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X))$,

$$\nabla^{\Lambda^{0,\bullet}}\alpha = \nabla^{\Lambda(T^*X)}\alpha. \tag{1.2.7}$$

Note the important fact that $\nabla^{\Lambda(T^*X)}$ preserves the bi-grading on $\Lambda^{\bullet,\bullet}(T^*X)$.

Let $\nabla^{\Lambda^{0,\bullet}\otimes E} := \nabla^{\Lambda^{0,\bullet}} \otimes 1 + 1 \otimes \nabla^E$ be the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^E

Proposition 1.2.1. *On $\Omega^{0,\bullet}(X,E)$, we have:*

$$\begin{aligned} \bar{\partial}^E &= \overline{w}^j \wedge \nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_j}, \\ \bar{\partial}^{E,*} &= -i_{\overline{w}_j} \nabla^{\Lambda^{0,\bullet}\otimes E}_{w_j}. \end{aligned} \tag{1.2.8}$$

Proof. We still denote by ∇^E the extension of the connection ∇^E to $\Omega^{\bullet,\bullet}(X,E)$ by the usual formula $\nabla^E(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg \alpha}\alpha \wedge \nabla^E s$ for $s \in \mathscr{C}^\infty(X,E)$ and $\alpha \in \mathscr{C}^\infty(X, \Lambda(T^*X))$ homogeneous. We know that $d = \varepsilon \circ \nabla^{\Lambda(T^*X)}$ where ε is the exterior multiplication (see [46, (1.2.44)]), so we get that $\nabla^E = \varepsilon \circ \nabla^{\Lambda(T^*X)\otimes E}$. Using (1.2.7), it follows that

$$\bar{\partial}^E = (\nabla^E)^{0,1} = \overline{w}^j \wedge \nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_j},$$

which is the first part of (1.2.8).

The second part of our proposition follows classically from the first by exactly the same computation as in [46, Lemma 1.4.4]. □

Definition 1.2.2. Let $v = v^{1,0} + v^{0,1} \in TX = T^{(1,0)}X \oplus T^{(0,1)}X$, and $\bar{v}^{(0,1),*} \in T^{*(0,1)}X$ the dual of $v^{1,0}$ for $\langle \cdot, \cdot \rangle$. We define the *Clifford action of TX on* $\Lambda^{0,\bullet}(T^*X)$ by

$$c(v) = \sqrt{2}\left(\bar{v}^{(0,1),*} \wedge -i_{v^{0,1}}\right). \tag{1.2.9}$$

We verify easily that for $u, v \in TX$,

$$c(u)c(v) + c(v)c(u) = -2\langle u, v\rangle, \tag{1.2.10}$$

and that for any skew-adjoint endomorphism A of TX,

$$\begin{aligned}\frac{1}{4}\langle Ae_i, e_j\rangle c(e_i)c(e_j) = -\frac{1}{2}\langle Aw_j, \overline{w}_j\rangle + \langle Aw_\ell, \overline{w}_m\rangle \overline{w}^m \wedge i_{\overline{w}_\ell} \\ + \frac{1}{2}\langle Aw_\ell, w_m\rangle i_{\overline{w}_\ell} i_{\overline{w}_m} + \frac{1}{2}\langle A\overline{w}_\ell, \overline{w}_m\rangle \overline{w}^\ell \wedge \overline{w}^m \wedge . \end{aligned}\tag{1.2.11}$$

Let $\nabla^{\det}$ be the Chern connection of $\det(T^{(1,0)}X) := \Lambda^n(T^{(1,0)}X)$, and ∇^{Cl} the Clifford connection on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX} and $\nabla^{\det}$ (see [46, (1.3.5)]). We also denote by ∇^{Cl} the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by ∇^{Cl} and ∇^E. By [46, (1.3.5)], (1.2.11) and the fact that $\nabla^{\det}$ is holomorphic, we get

$$\nabla^{\mathrm{Cl}} = \nabla^{\Lambda^{0,\bullet}}. \tag{1.2.12}$$

Let $D^{c,E}$ be the associated spin^c Dirac operator:

$$D^{c,E} = \sum_{j=1}^{2n} c(e_j)\nabla^{\mathrm{Cl}}_{e_j} \colon \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet}(X, E). \tag{1.2.13}$$

By (1.2.8) and (1.2.12), we have

Theorem 1.2.3. *D^E is equal to the spin^c Dirac operator $D^{c,E}$ acting on $\Omega^{0,\bullet}(X, E)$.*

Remark 1.2.4. Note that all the results proved in the beginning of this subsection hold without assuming the pre-quantization condition (1.1.1), but from now on we will use it.

Let (F, h^F) be a Hermitian vector bundle on X and let ∇^F be a Hermitian connection on F. Then the *Bochner Laplacian* Δ^F acting on $\mathscr{C}^\infty(X, F)$ is defined by

$$\Delta^F = -\sum_{j=1}^{2n}\left((\nabla^F_{e_j})^2 - \nabla^F_{\nabla^{TX}_{e_j} e_j}\right). \tag{1.2.14}$$

On $\Omega^{0,\bullet}(X)$, we define the *number operator* $\mathscr{N}$ by

$$\mathscr{N}|_{\Omega^{0,j}(X)} = j, \tag{1.2.15}$$

and we also denote by $\mathscr{N}$ the operator $\mathscr{N} \otimes 1$ acting on $\Omega^{0,\bullet}(X, F)$.

The bundle L^p is endowed with the connection ∇^{L^p} induced by ∇^L (which is also its Chern connection). Let $\nabla^{L^p\otimes E} := \nabla^{L^p} \otimes 1 + 1 \otimes \nabla^E$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. We will denote

$$D_p = D^{L^p\otimes E}. \tag{1.2.16}$$

Theorem 1.2.5. *The square of D_p is given by*

$$\begin{aligned}D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} - R^E(w_j, \overline{w}_j) - 2\pi pn + 4\pi p\mathscr{N} + 2\left(R^E + \frac{1}{2}R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell} \\ + R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} + R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m. \end{aligned}\tag{1.2.17}$$

Proof. By Theorem 1.2.3, we can use [46, Theorem 1.3.5]:

$$D_p^2 = \Delta^{\mathrm{Cl}} + \frac{r^X}{4} + \frac{1}{2}\left(R^{L^p\otimes E} + \frac{1}{2}R^{\det}\right)(e_i,e_j)c(e_i)c(e_j), \tag{1.2.18}$$

where r^X is the scalar curvature of X. From (1.2.12), we see that $\Delta^{\mathrm{Cl}} = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E}$. Moreover, $r^X = 2R^{\det}(w_j,\overline{w}_j)$ and $R^{L^p\otimes E} = R^E + pR^L$. Using the equivalent of (1.2.11) for 2-forms (substituting $A(\cdot,\cdot)$ for $\langle A\cdot,\cdot\rangle$) and the fact that R^L and $R^{\det}$ are $(1,1)$-forms, (1.2.18) reads

$$\begin{aligned} D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} &+ \frac{1}{2}R^{\det}(w_j,\overline{w}_j) - \left(R^E(w_j,\overline{w}_j) + pR^L(w_j,\overline{w}_j) + \frac{1}{2}R^{\det}(w_j,\overline{w}_j)\right)\\ &+ 2\left(R^E + pR^L + \frac{1}{2}R^{\det}\right)(w_\ell,\overline{w}_m)\overline{w}^m\wedge i_{\overline{w}_\ell} + R^E(w_\ell,w_m)i_{\overline{w}_\ell}i_{\overline{w}_m}\\ &+ R^E(\overline{w}_\ell,\overline{w}_m)\overline{w}^\ell\wedge\overline{w}^m. \end{aligned}$$

Thanks to (1.1.1), we have $R^L(w_\ell,\overline{w}_m) = 2\pi\delta_{\ell m}$. Moreover, $\mathcal{N} = \sum_\ell \overline{w}^\ell\wedge i_{\overline{w}_\ell}$, thus we get Theorem 1.2.5. □

1.2.2 Rescaling D_p^2

In this subsection, we rescale D_p^2, but to do this we must define it on a vector space. Therefore, we will use normal coordinates to transfer the problem on the tangent space to X at a fixed point. Then we give a Taylor expansion of the rescaled operator, but the problem is that each operator acts on a different space, namely

$$\mathbb{E}_p := \Lambda^{0,\bullet}(T^*X)\otimes L^p\otimes E,$$

so we must first handle this issue.

Fix $x_0\in X$. For the rest of this chapter, we fix $\{w_j\}$ an orthonormal basis of $T^{(1,0)}_{x_0}X$, with dual basis $\{w^j\}$, and we construct an orthonormal basis $\{e_i\}$ of $T_{x_0}X$ from $\{w_j\}$ as in (1.2.1).

For $\varepsilon>0$, we denote by $B^X(x_0,\varepsilon)$ and $B^{T_{x_0}X}(0,\varepsilon)$ the open balls in X and $T_{x_0}X$ with center x_0 and 0 and radius ε respectively. If $\exp^X_{x_0}$ is the Riemannian exponential of X, then for ε small enough, $Z\in B^{T_{x_0}X}(0,\varepsilon)\mapsto \exp^X_{x_0}(Z)\in B^X(x_0,\varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}X$ with $\mathbb{R}^{2n}$ via the orthonormal basis $\{e_i\}$:

$$(Z_1,\dots,Z_{2n})\in\mathbb{R}^{2n}\mapsto \sum_i Z_ie_i\in T_{x_0}X. \tag{1.2.19}$$

From now on, we will always identify $B^{T_{x_0}X}(0,\varepsilon)$ and $B^X(x_0,\varepsilon)$. Note that in this identification, the radial vector field $\mathcal{R}=\sum_i Z_ie_i$ becomes $\mathcal{R}=Z$, so Z can be viewed as a point or as a tangent vector.

For $Z\in B^{T_{x_0}X}(0,\varepsilon)$, we identify (L_Z,h^L_Z), (E_Z,h^E_Z) and $(\Lambda^{0,\bullet}_Z(T^*X),h^{\Lambda^{0,\bullet}}_Z)$ with $(L_{x_0},h^L_{x_0})$, $(E_{x_0},h^E_{x_0})$ and $(\Lambda^{0,\bullet}(T^*_{x_0}X),h^{\Lambda^{0,\bullet}}_{x_0})$ by parallel transport with respect to the connection ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$ along the geodesic ray $t\in[0,1]\mapsto tZ$. We denote by Γ^L, Γ^E and $\Gamma^{\Lambda^{0,\bullet}}$ the corresponding connection forms of ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$.

Remark 1.2.6. Note that since $\nabla^{\Lambda^{0,\bullet}}$ preserves the degree, the identification between $\Lambda^{0,\bullet}(T^*X)$ and $\Lambda^{0,\bullet}(T^*_{x_0}X)$ is compatible with the degree. Thus, $\Gamma^{\Lambda^{0,\bullet}}_Z\in\bigoplus_j \mathrm{End}(\Lambda^{0,j}(T^*X))$.

Let S_L be a unit vector of L_{x_0}. It gives an isometry $L^p_{x_0} \simeq \mathbb{C}$, which induces an isometry

$$\mathbb{E}_{p,x_0} \simeq (\Lambda^{0,\bullet}(T^*X) \otimes E)_{x_0} =: \mathbb{E}_{x_0}. \tag{1.2.20}$$

Thus, in our trivialization, D_p^2 acts on $\mathbb{E}_{x_0}$, but this action may *a priori* depend on the choice of S_L. In fact, since the operator D_p^2 takes values in $\mathrm{End}(\mathbb{E}_{p,x_0})$ which is canonically isomorphic to $\mathrm{End}(\mathbb{E})_{x_0}$ (by the natural identification $\mathrm{End}(L^p) \simeq \mathbb{C}$), all our formulas do not depend on this choice.

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$, and $\kappa(Z)$ be the smooth positive function defined for $|Z| \le \varepsilon$ by

$$dv_X(Z) = \kappa(Z) dv_{TX}(Z), \tag{1.2.21}$$

with $\kappa(0) = 1$.

Definition 1.2.7. We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_0}X$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}
&(S_t s)(Z) = s(Z/t),\\
&\nabla_t = t S_t^{-1} \kappa^{1/2} \nabla^{\mathrm{Cl}_0} \kappa^{-1/2} S_t,\\
&\nabla_0 = \nabla + \frac{1}{2} R^L_{x_0}(Z, \cdot),\\
&\mathscr{L}_t = t^2 S_t^{-1} \kappa^{1/2} D_p^2 \kappa^{-1/2} S_t,\\
&\mathscr{L}_0 = -\sum_i (\nabla_{0,e_i})^2 + 4\pi\mathscr{N} - 2\pi n.
\end{aligned} \tag{1.2.22}$$

Let $||\cdot||_{L^2}$ be the L^2-norm induced by $h^{\mathbb{E}_{x_0}}$ and dv_{TX}. We can now state the key result in our approach to Theorems 1.1.4 and 1.1.7:

Theorem 1.2.8. *There exist second-order formally self-adjoint (with respect to $||\cdot||_{L^2}$) differential operators $\mathcal{O}_r$ with polynomial coefficients such that for all $m \in \mathbb{N}$,*

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^m t^r \mathcal{O}_r + O(t^{m+1}). \tag{1.2.23}$$

Furthermore, each $\mathcal{O}_r$ can be decomposed as

$$\mathcal{O}_r = \mathcal{O}_r^0 + \mathcal{O}_r^{+2} + \mathcal{O}_r^{-2}, \tag{1.2.24}$$

where $\mathcal{O}_r^k$ changes the degree of the form it acts on by k.

Proof. The first part of the theorem (i.e., equation (1.2.23)) is contained in [47, Theorem 1.4]. We will briefly recall how they obtained this result.

Let Φ_E be the smooth self-adjoint section of $\mathrm{End}(\mathbb{E}_{x_0})$ on $B^{T_{x_0}X}(0,\varepsilon)$:

$$\begin{aligned}
\Phi_E = -R^E(w_j, \overline{w}_j) + 2\left(R^E + \frac{1}{2} R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell}\\
+ R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} + R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m.
\end{aligned} \tag{1.2.25}$$

We can see that we can decompose $\Phi_E = \Phi_E^0 + \Phi_E^{+2} + \Phi_E^{-2}$, where

$$\begin{aligned}
&\Phi_E^0 = R^E(w_j, \overline{w}_j) + 2\left(R^E + \frac{1}{2} R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell} \text{ preserves the degree},\\
&\Phi_E^{+2} = R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m \text{ rises the degree by 2},\\
&\Phi_E^{-2} = R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} \text{ lowers the degree by 2}.
\end{aligned} \tag{1.2.26}$$

Using Theorem 1.2.5, we find that :

$$D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} + p(-2\pi n + 4\pi\mathcal{N}) + \Phi_E. \tag{1.2.27}$$

Let $g_{ij}(Z) = g^{TX}(e_i,e_j)(Z)$ and $\left(g^{ij}(Z)\right)_{ij}$ be the inverse of the matrix $\left(g_{ij}(Z)\right)_{ij}$. Let $\left(\nabla^{TX}_{e_i}e_j\right)(Z) = \Gamma_{ij}^k(Z)e_k$. As in [46, (4.1.34)], by (1.2.22) and (1.2.27), we get:

$$\begin{aligned} \nabla_{t,\cdot} &= \kappa^{1/2}(tZ)\left(\nabla_\cdot + t\Gamma^{\Lambda^{0,\bullet}}_{tZ} + \frac{1}{t}\Gamma^L_{tZ} + t\Gamma^E_{tZ}\right)\kappa^{-1/2}(tZ),\\ \mathscr{L}_t &= -g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma_{ij}^k(tZ)\nabla_{t,e_k}\right) - 2\pi n + 4\pi\mathcal{N} + t^2\Phi_E(tZ). \end{aligned} \tag{1.2.28}$$

Moreover, $\kappa = (\det(g_{ij}))^{1/2}$, thus we can prove equations (1.2.23) as in [46, Theorem 4.1.7] by taking the Taylor expansion of each term appearing in (1.2.28). Note that in [46], every data has to be extended to $T_{x_0}X$ to make the analysis work, but as we admit the result, we do not have to worry about it and simply restrict ourselves to a neighborhood of x_0.

Now, it is clear that in the formula for $\mathscr{L}_t$ in (1.2.28), the term

$$\mathscr{L}_t^0 := -g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma_{ij}^k(tZ)\nabla_{t,e_k}\right) - 2\pi n + 4\pi\mathcal{N} + t^2\Phi_E^0(tZ) \tag{1.2.29}$$

preserves the degree, because $\Gamma^{\Lambda^{0,\bullet}}$ does (as explained in Remark 1.2.6). Thus, using (1.2.26) and taking Taylor expansion of $\mathscr{L}_t$ in (1.2.28), we can write

$$\begin{aligned} \mathscr{L}_t^0 &= \mathscr{L}_0 + \sum_{r=1}^{\infty} t^r\mathcal{O}_r^0,\\ t^2\Phi_E^{\pm2}(tZ) &= \sum_{r=2}^{\infty} t^r\mathcal{O}_r^{\pm2}. \end{aligned} \tag{1.2.30}$$

From (1.2.30), we get (1.2.24).

Finally, due to the presence of the conjugation by $\kappa^{1/2}$ in (1.2.22), $\mathscr{L}_t$ is a formally self-adjoint operator on $\mathscr{C}^\infty(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ with respect to $||\cdot||_{L^2}$. Thus, $\mathscr{L}_0$ and the $\mathcal{O}_r$'s also are. □

Recall that $\mathscr{R} = (R^E)^{0,2} \in \Omega^{0,2}(X,\mathrm{End}(E))$.

Proposition 1.2.9. *We have*

$$\mathcal{O}_1 = 0. \tag{1.2.31}$$

For $\mathcal{O}_2$, we have the formulas:

$$\mathcal{O}_2^{+2} = \mathscr{R}_{x_0}\,, \quad \mathcal{O}_2^{-2} = \left(\mathscr{R}_{x_0}\right)^*, \tag{1.2.32}$$

and

$$\begin{aligned} \mathcal{O}_2^0 = \frac{1}{3}\langle R^{TX}_{x_0}(Z,e_i)Z,e_j\rangle\nabla_{0,e_i}\nabla_{0,e_j} - R^E_{x_0}(w_j,\overline{w}_j) - \frac{r^X_{x_0}}{6}\\ + \left(\left\langle\frac{1}{3}R^{TX}_{x_0}(Z,e_k)e_k + \frac{\pi}{3}R^{TX}_{x_0}(z,\bar z)Z,e_j\right\rangle - R^E_{x_0}(Z,e_j)\right)\nabla_{0,e_j}. \end{aligned} \tag{1.2.33}$$

Proof. For $F = L, E$ or $\Lambda^{0,\bullet}(T^*X)$, it is known that (see for instance [46, Lemma 1.2.4])

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^F)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^F)_{x_0}(Z,e_j)\frac{Z^\alpha}{\alpha!}, \tag{1.2.34}$$

and in particular,

$$\Gamma_Z^F(e_j) = \frac{1}{2} R_{x_0}^F(Z, e_j) + O(|Z|^2). \tag{1.2.35}$$

Furthermore, we know that

$$g_{ij}(Z) = \delta_{ij} + O(|Z|^2) : \tag{1.2.36}$$

it is the Gauss lemma (see [46, (1.2.19)]). It implies that

$$\kappa(Z) = |\det(g_{ij}(Z))|^{1/2} = 1 + O(|Z|^2). \tag{1.2.37}$$

Moreover, the second line of [46, (4.1.103)] entails

$$\frac{\sqrt{-1}}{2\pi} R_Z^L(Z, e_j) = \langle JZ, e_j \rangle + O(|Z|^3), \tag{1.2.38}$$

and thus by (1.2.34) and (1.2.38)

$$\Gamma_Z^L = \frac{1}{2} R_{x_0}^L(Z, e_j) + O(|Z|^3). \tag{1.2.39}$$

Using (1.2.28), (1.2.35), (1.2.37) and (1.2.39), we see that

$$\nabla_t = \nabla_0 + O(t^2). \tag{1.2.40}$$

Finally, using again (1.2.28), (1.2.36) and (1.2.40), we get $\mathscr{O}_1 = 0$.

Concerning $\mathscr{O}_2^{\pm 2}$, from (1.2.30), we see that

$$\begin{aligned}
\mathscr{O}_2^{+2} &= \Phi_E^{+2}(0) = R_{x_0}^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m = (R_{x_0}^E)^{0,2} = \mathscr{R}_{x_0}, \\
\mathscr{O}_2^{-2} &= \Phi_E^{-2}(0) = R_{x_0}^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} = \Big((R_{x_0}^E)^{0,2}\Big)^* = (\mathscr{R}_{x_0})^* .
\end{aligned} \tag{1.2.41}$$

Finally, by (1.2.29) and [46, (4.1.34)], we see that our $\mathscr{L}_t^0$ corresponds to $\mathscr{L}_t$ in [46]. Thus, by (1.2.30) and [46, (4.1.31)], our $\mathscr{O}_2^0$ is equal to their $\mathscr{O}_2$ (this is because in their case, E is holomorphic, so R^E is a $(1,1)$-form and there is no term changing the degree in $(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*})^2$, but the terms preserving the degree are the same as ours). Hence (1.2.33) follows from [46, Theorem 4.1.25]. □

1.2.3 Bergman kernel of the limit operator $\mathscr{L}_0$

In this subsection, we study more precisely the operator $\mathscr{L}_0$.

We introduce the complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus, we get $Z = z + \bar z$, $w_j = \sqrt{2}\frac{\partial}{\partial z_j}$ and $\overline{w}_j = \sqrt{2}\frac{\partial}{\partial \bar z_j}$. We will identify z to $\sum_j z_j \frac{\partial}{\partial z_j}$ and $\bar z$ to $\sum_j \bar z_j \frac{\partial}{\partial \bar z_j}$ when we consider z and $\bar z$ as vector fields.

Set

$$\begin{aligned}
b_j &= -2\nabla_{0,\frac{\partial}{\partial z_j}}, & b_j^+ &= 2\nabla_{0,\frac{\partial}{\partial \bar z_j}}, \\
b &= (b_1, \dots, b_n), & \mathscr{L} &= -\sum_i (\nabla_{0,e_i})^2 - 2\pi n.
\end{aligned} \tag{1.2.42}$$

By definition, $\nabla_0 = \nabla + \frac{1}{2} R_{x_0}^L(Z, \cdot)$ so we get

$$b_i = -2\frac{\partial}{\partial z_i} + \pi \bar z_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar z_i} + \pi z_i, \tag{1.2.43}$$

and for any polynomial $g(z,\bar z)$ in z and $\bar z$,

$$\begin{aligned} [b_i, b_j^+] &= -4\pi\delta_{ij}, & [b_i, b_j] &= [b_i^+, b_j^+] = 0, \\ [g(z,\bar z), b_j] &= 2\tfrac{\partial}{\partial z_j} g(z,\bar z), & [g(z,\bar z), b_j^+] &= -2\tfrac{\partial}{\partial \bar z_j} g(z,\bar z). \end{aligned} \tag{1.2.44}$$

Finally, a simple calculation shows:

$$\mathscr{L} = \sum_i b_i b_i^+ \text{ and } \mathscr{L}_0 = \mathscr{L} + 4\pi\mathcal{N}. \tag{1.2.45}$$

Recall that we denoted by $||\cdot||_{L^2}$ the L^2-norm associated with $h^{\mathbb{E}_{x_0}}$ and dv_{TX}. As for this norm $b_i^+ = (b_i)^*$, we see that $\mathscr{L}$ and $\mathscr{L}_0$ are self-adjoint with respect to this norm.

The next theorem is proved in [46, Theorem 4.1.20]:

Theorem 1.2.10. *The spectrum of the restriction of $\mathscr{L}$ to $L^2(\mathbb{R}^{2n})$ is $\mathrm{Sp}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = 4\pi\mathbb{N}$ and an orthogonal basis of the eigenspace for the eigenvalue $4\pi k$ is*

$$b^\alpha \left(z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right)\right), \text{ with } \alpha,\beta\in\mathbb{N}^n \text{ and } \sum_i \alpha_i = k. \tag{1.2.46}$$

Especially, an orthonormal basis of $\ker(\mathscr{L}|_{L^2(\mathbb{R}^{2n})})$ is

$$\left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right), \tag{1.2.47}$$

and thus if $\mathscr{P}(Z,Z')$ is the smooth kernel of $\mathscr{P}$ the orthogonal projection from $(L^2(\mathbb{R}^{2n}), ||\cdot||_0)$ onto $\ker(\mathscr{L})$ (where $||\cdot||_0$ is the L^2-norm associated to $g^{TX}_{x_0}$) with respect to $dv_{TX}(Z')$, we have

$$\mathscr{P}(Z,Z') = \exp\left(-\frac{\pi}{2}(|z|^2 + |z'|^2 - 2z\cdot\bar z')\right). \tag{1.2.48}$$

Now let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}), ||\cdot||_{L^2})$ onto $N := \ker(\mathscr{L}_0)$, and $P^N(Z,Z')$ be its smooth kernel with respect to $dv_{TX}(Z')$. From (1.2.45), we have:

$$P^N(Z,Z') = \mathscr{P}(Z,Z') I_0. \tag{1.2.49}$$

1.3 The first coefficient in the asymptotic expansion

In this section we prove Theorem 1.1.4. We will proceed as follows.

In Section 1.3.1, following [46, Section 4.1.7], we will give a formula for $\boldsymbol{b}_r$ involving the $\mathcal{O}_k$'s and $\mathscr{L}_0$.

In Section 1.3.2, we will see how this formula entails Theorem 1.1.4.

1.3.1 A formula for $\boldsymbol{b}_r$

By Theorem 1.2.10 and (1.2.45), we know that for every $\lambda\in\delta$ the unit circle in $\mathbb{C}$, $(\lambda-\mathscr{L}_0)^{-1}$ exists.

Let $f(\lambda,t)$ be a formal power series on t with values in $\mathrm{End}(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}))$:

$$f(\lambda,t) = \sum_{r=0}^{+\infty} t^r f_r(\lambda) \text{ with } f_r(\lambda)\in\mathrm{End}(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})). \tag{1.3.1}$$

Consider the equation of formal power series on t for $\lambda \in \delta$:

$$\left(\lambda - \mathscr{L}_0 - \sum_{r=1}^{+\infty} t^r \mathcal{O}_r\right) f(\lambda, t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})}. \tag{1.3.2}$$

We then find that

$$\begin{aligned} f_0(\lambda) &= (\lambda - \mathscr{L}_0)^{-1}, \\ f_r(\lambda) &= (\lambda - \mathscr{L}_0)^{-1} \sum_{j=1}^{r} \mathcal{O}_j f_{r-j}(\lambda). \end{aligned} \tag{1.3.3}$$

Thus by (1.2.31) and by induction,

$$f_r(\lambda) = \left(\sum_{\substack{r_1 + \cdots + r_k = r \\ r_j \geq 2}} (\lambda - \mathscr{L}_0)^{-1} \mathcal{O}_{r_1} \ldots (\lambda - \mathscr{L}_0)^{-1} \mathcal{O}_{r_k} \right) (\lambda - \mathscr{L}_0)^{-1}. \tag{1.3.4}$$

Definition 1.3.1. Following [46, (4.1.91)], we define $\mathscr{F}_r$ by

$$\mathscr{F}_r = \frac{1}{2\pi\sqrt{-1}} \int_\delta f_r(\lambda) d\lambda, \tag{1.3.5}$$

and we denote by $\mathscr{F}_r(Z, Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

Theorem 1.3.2. *The following equation holds:*

$$\boldsymbol{b}_r(x_0) = \mathscr{F}_{2r}(0, 0). \tag{1.3.6}$$

Proof. This formula follows from [46, Theorem 8.1.4], as [46, (4.1.97)] follows from [46, Theorem 4.1.24], remembering that in our situation, the Bergman kernel P_p is not supported in degree 0. □

1.3.2 Proof of Theorem 1.1.4

Let $T_{\mathbf{r}}(\lambda) = (\lambda - \mathscr{L}_0)^{-1} \mathcal{O}_{r_1} \ldots (\lambda - \mathscr{L}_0)^{-1} \mathcal{O}_{r_k} (\lambda - \mathscr{L}_0)^{-1}$ be the term in the sum (1.3.4) corresponding to $\mathbf{r} = (r_1, \ldots, r_k)$. Let $N^\perp$ be the orthogonal of N in $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and $P^{N^\perp}$ be the associated orthogonal projector. In $T_{\mathbf{r}}(\lambda)$, each term $(\lambda - \mathscr{L}_0)^{-1}$ can be decomposed as

$$(\lambda - \mathscr{L}_0)^{-1} = (\lambda - \mathscr{L}_0)^{-1} P^{N^\perp} + \frac{1}{\lambda} P^N. \tag{1.3.7}$$

Set

$$L^{N^\perp}(\lambda) = (\lambda - \mathscr{L}_0)^{-1} P^{N^\perp}, \qquad L^N(\lambda) = \frac{1}{\lambda} P^N. \tag{1.3.8}$$

By (1.2.45), $\mathscr{L}_0$ preserves the degree, and thus so do $(\lambda - \mathscr{L}_0)^{-1}$, $L^{N^\perp}$ and L^N.

For $\eta = (\eta_1, \ldots, \eta_{k+1}) \in \{N, N^\perp\}^{k+1}$, let

$$T^\eta_{\mathbf{r}}(\lambda) = L^{\eta_1}(\lambda) \mathcal{O}_{r_1} \ldots L^{\eta_k}(\lambda) \mathcal{O}_{r_k} L^{\eta_{k+1}}(\lambda). \tag{1.3.9}$$

We can decompose:

$$T_{\mathbf{r}}(\lambda) = \sum_{\eta = (\eta_1, \ldots, \eta_{k+1})} T^\eta_{\mathbf{r}}(\lambda), \tag{1.3.10}$$

and by (1.3.4) and (1.3.5)

$$\mathscr{F}_{2r} = \frac{1}{2\pi\sqrt{-1}} \sum_{\substack{r_1+\cdots+r_k=2r \\ (\eta_1,\dots,\eta_{k+1})}} \int_\delta T_{\mathbf{r}}^{\eta}(\lambda)d\lambda. \tag{1.3.11}$$

Note that $L^{N^\perp}(\lambda)$ is an holomorphic function of λ, so

$$\int_\delta L^{N^\perp}(\lambda)\mathscr{O}_{r_1}\dots L^{N^\perp}(\lambda)\mathscr{O}_{r_k}L^{N^\perp}(\lambda)d\lambda = 0. \tag{1.3.12}$$

Thus, in (1.3.11), every non-zero term that appears contains at least one $L^N(\lambda)$:

$$\int_\delta T_{\mathbf{r}}^{\eta}(\lambda)d\lambda \neq 0 \Rightarrow \text{there exists } i_0 \text{ such that } \eta_{i_0} = N. \tag{1.3.13}$$

Now fix k and j in $\mathbb{N}$. Let $s \in L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ be a form of degree $2j$, $\mathbf{r} \in (\mathbb{N}\backslash\{0,1\})^k$ such that $\sum_i r_i = 2r$ and $\eta = (\eta_1, \dots, \eta_{k+1}) \in \{N, N^\perp\}^{k+1}$ such that there is a i_0 satisfying $\eta_{i_0} = N$. We want to find a necessary condition for $I_{2j}T_{\mathbf{r}}^{\eta}(\lambda)I_{2j}s$ to be non-zero.

Suppose then that $I_{2j}T_{\mathbf{r}}^{\eta}(\lambda)I_{2j}s \neq 0$. Since $L^{\eta_{i_0}} = \frac{1}{\lambda}P^N$, and N is concentrated in degree 0, we must have

$$\deg\Big(\mathscr{O}_{r_{i_0}}L^{\eta_{i_0+1}}(\lambda)\mathscr{O}_{r_{i_0+1}}\dots L^{\eta_k}(\lambda)\mathscr{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s\Big) = 0,$$

but each $L^{\eta_i}(\lambda)$ preserves the degree, and by Theorem 1.2.8 each $\mathscr{O}_{r_i}$ lowers the degree at most by 2, so

$$0 = \deg\Big(\mathscr{O}_{r_{i_0}}L^{\eta_{i_0+1}}(\lambda)\mathscr{O}_{r_{i_0+1}}\dots L^{\eta_k}(\lambda)\mathscr{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s\Big) \geq 2j - 2(k - i_0 + 1),$$

and thus

$$2j \leq 2(k - i_0 + 1). \tag{1.3.14}$$

Similarly, $L^{\eta_1}(\lambda)\mathscr{O}_{r_1}\dots L^{\eta_k}(\lambda)\mathscr{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s$ must have a non-zero component in degree $2j$ and by Theorem 1.2.8 each $\mathscr{O}_{r_i}$ rises the degree at most by 2, so $2j$ must be less or equal to the number of $\mathscr{O}_{r_i}$'s appearing before $\mathscr{O}_{r_{i_0}}$, that is

$$2j \leq 2(i_0 - 1). \tag{1.3.15}$$

With (1.3.14) and (1.3.15), we find

$$4j \leq 2k. \tag{1.3.16}$$

Finally, since for every i, $r_i \geq 2$ and $\sum_{i=1}^k r_i = 2r$, we have $2k \leq 2r$, and thus

$$4j \leq 2k \leq 2r. \tag{1.3.17}$$

Consequently, if $r < 2j$ we have $I_{2j}T_{\mathbf{r}}^{\eta}(\lambda)I_{2j} = 0$, and by (1.3.11), we find $I_{2j}\mathscr{F}_{2r}I_{2j} = 0$. Using Theorem 1.3.2, we find

$$I_{2j}\boldsymbol{b}_r I_{2j} = 0,$$

which, combined with Theorem 1.1.3, entails the first part of Theorem 1.1.4.

For the second part of this theorem, let us assume that we are in the limit case where $r = 2j$. We also suppose that $j \geq 1$, because in the case $j = 0$, [46, (8.1.5)] implies that $\boldsymbol{b}_0(x_0) = \mathscr{F}_0(0,0) = I_0\mathscr{P}(0,0) = I_0$, so Theorem 1.1.4 is true for $j = 0$.

In $I_{2j}\mathscr{F}_{4j}I_{2j}$, there is only one term satisfying equations (1.3.14), (1.3.15) and (1.3.17): first we see that (1.3.17) imply that $r = k = 2j$ and for all i, $r_i = 2$, while (1.3.14) and

(1.3.15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j. Moreover, since the degree must decrease by $2j$ and then increase by $2j$ with only $k = 2j$ $\mathcal{O}_{r_i}$'s available, only $\mathcal{O}_2^{+2}$ and $\mathcal{O}_2^{-2}$ appear in $I_{2j}\mathscr{F}_{4j}I_{2j}$, and not $\mathcal{O}_2^{0}$. To summarize:

$$\begin{aligned}
I_{2j}\mathscr{F}_{4j}I_{2j} &= \frac{1}{2\pi\sqrt{-1}}\int_\delta I_{2j}\left((\lambda-\mathscr{L}_0)^{-1}P^{N^\perp}\mathcal{O}_2^{+2}\right)^j \frac{1}{\lambda}P^N\left(\mathcal{O}_2^{-2}(\lambda-\mathscr{L}_0)^{-1}P^{N^\perp}\right)^j I_{2j}d\lambda \\
&= I_{2j}\left(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2^{+2}\right)^j P^N\left(\mathcal{O}_2^{-2}\mathscr{L}_0^{-1}P^{N^\perp}\right)^j I_{2j} \\
&= I_{2j}\left(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2}\right)^j P^N\left(\mathcal{O}_2^{-2}\mathscr{L}_0^{-1}\right)^j I_{2j},
\end{aligned} \tag{1.3.18}$$

because by (1.2.45), $L^2(\mathbb{R}^{2n}, (\Lambda^{0,>0}(T^*X)\otimes E)_{x_0}) \subset N^\perp$, so we can remove the $P^{N^\perp}$'s.

Let $A = I_{2j}\left(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2}\right)^j P^N$. Since $(\mathcal{O}_2^{+2})^* = \mathcal{O}_2^{-2}$ (see Proposition 1.2.9) and $\mathscr{L}_0$ is self-adjoint, the adjoint of A is $A^* = P^N\left(\mathcal{O}_2^{-2}\mathscr{L}_0^{-1}\right)^j I_{2j}$, and thus

$$I_{2j}\mathscr{F}_{4j}I_{2j} = AA^*. \tag{1.3.19}$$

Recall that $P^N = \mathscr{P}I_0$ (see (1.2.49)). Let $s \in L^2(\mathbb{R}^{2n}, E_{x_0})$, since $\mathscr{L}_0 = \mathscr{L} + 4\pi\mathscr{N}$ and $\mathscr{L}\mathscr{P}s = 0$, the term $(\mathscr{P}s)\mathscr{R}_{x_0}$ is an eigenfunction of $\mathscr{L}_0$ for the eigenvalue $2\times 4\pi$. Thus, we get

$$\mathscr{L}_0^{-1}\mathcal{O}_2^{+2}P^N s = \mathscr{L}_0^{-1}\mathcal{O}_2^{+2}\mathscr{P}s = \mathscr{L}_0^{-1}\left((\mathscr{P}s)\mathscr{R}_{x_0}\right) = \frac{1}{4\pi}\frac{1}{2}\mathscr{R}_{x_0}\mathscr{P}s. \tag{1.3.20}$$

Now, an easy induction shows that

$$A = \frac{1}{(4\pi)^j}\frac{1}{2\times 4\times\cdots\times 2j}I_{2j}\mathscr{R}_{x_0}^j\mathscr{P} = \frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}_{x_0}^j\mathscr{P}. \tag{1.3.21}$$

Let $A(Z,Z')$ and $A^*(Z,Z')$ be the smooth kernels of A and A^* with respect to $dv_{TX}(Z')$. By (1.3.19), $I_{2j}\mathscr{F}_{4j}I_{2j}(0,0) = \int_{\mathbb{R}^{2n}} A(0,Z)A^*(Z,0)dZ$. Thanks to

$$\int_{\mathbb{R}^{2n}} \mathscr{P}(0,Z)\mathscr{P}(Z,0)dZ = (\mathscr{P}\circ\mathscr{P})(0,0) = \mathscr{P}(0,0) = 1 \tag{1.3.22}$$

and (1.3.21), we find (1.1.9).

1.4 The second coefficient in the asymptotic expansion

In this section, we prove Theorem 1.1.7. Using (1.3.6), we know that

$$I_{2j}\boldsymbol{b}_{2j+1}I_{2j}(0,0) = I_{2j}\mathscr{F}_{4j+2}I_{2j}(0,0). \tag{1.4.1}$$

In Section 1.4.1, we decompose this term into three terms, and then in Sections 1.4.2 and 1.4.3 we handle them separately.

For the rest of the section we fix $j \in [\![0,n]\!]$. For every smoothing operator F acting on $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ that appears in this section, we will denote by $F(Z,Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

1.4.1 Decomposition of the problem

Applying inequality (1.3.17) with $r = 2j+1$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$, the non-zero terms $\int_\delta T^\eta_{\mathrm{r}}(\lambda)d\lambda$ appearing in decomposition (1.3.11) satisfy $k = 2j$ or $k = 2j+1$. Since $\sum_i r_i = 4j+2$ and $r_i \geq 2$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$ there are three types of terms $T^\eta_{\mathrm{r}}(\lambda)$ with non-zero integral, in which:

- for $k = 2j$:
  - there are $2j-2$ $\mathcal{O}_{r_i}$'s equal to $\mathcal{O}_2$ and 2 equal to $\mathcal{O}_3$: we will denote by I the sum of these terms,
  - there are $2j-1$ $\mathcal{O}_{r_i}$'s equal to $\mathcal{O}_2$ and 1 equal to $\mathcal{O}_4$: we will denote by II the sum of these terms,
- for $k = 2j+1$:
  - all the $\mathcal{O}_{r_i}$'s are equal to $\mathcal{O}_2$: we will denote by III the sum of these terms.

We thus have a decomposition

$$I_{2j}\mathscr{F}_{4j+2}I_{2j} = \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{1.4.2}$$

Remark 1.4.1. Note that for the two sums I and II to be non-zero, we must have $j \geq 1$. Moreover, in the two first cases, as $k = 2j$, by the same reasoning as in Section 1.3.2, (1.3.14) and (1.3.15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j, and that only $\mathcal{O}_2^{\pm 2}$, $\mathcal{O}_3^{\pm 2}$ and $\mathcal{O}_4^{\pm 2}$ appear in I and II, and not some $\mathcal{O}^0_{r_i}$.

1.4.2 The term involving only $\mathcal{O}_2$

Lemma 1.4.2. *In any term $T^\eta_{\mathrm{r}}(\lambda)$ appearing in the term* III *(with non-vanishing integral), the i_0 such that $\eta_{i_0} = N$ is unique and equal to j or $j+1$. If we denote by* III_a *and* III_b *the sum of the terms corresponding to these two cases, we have:*

$$\begin{aligned}
\mathrm{III}_a &= \sum_{k=0}^{j} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^0)(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^j I_{2j},\\
\mathrm{III}_b &= (\mathrm{III}_a)^*,\\
\mathrm{III} &= \mathrm{III}_a + \mathrm{III}_b.
\end{aligned} \tag{1.4.3}$$

Remark 1.4.3. For the same reason as for (1.3.18), we have removed the $P^{N^\perp}$'s in (1.4.3) without getting any problem with the existence of $\mathscr{L}_0{}^{-1}$.

Proof. Fix a term $T^\eta_{\mathrm{r}}(\lambda)$ appearing in the term III with non-vanishing integral. Using again the same reasoning as in Section 1.3.2, we see that there exists at most two indices i_0 such that $\eta_{i_0} = N$, and that they are in $\{j, j+1\}$. Indeed, with only $2j+1$ $\mathcal{O}_{r_i}$'s at our disposal, we need j of them before the first P^N, and j after the last one.

Now, the only possible term with $\eta_j = \eta_{j+1} = N$ is:

$$(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^j P^N \mathcal{O}_2^0 P^N (\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^j.$$

To prove that this term is vanishing, we will use [49]. By (1.2.33), [49, (3.13),(3.16b)] and [49, (4.1a)] we see that $\mathscr{P}\mathcal{O}_2^0\mathscr{P} = 0$, and so

$$P^N\mathcal{O}_2^0P^N = \mathscr{P}\mathcal{O}_2^0\mathscr{P}I_0 = 0, \tag{1.4.4}$$

we have proved the first part of the lemma.

The second part follows from the reasoning made at the beginning of this proof, and the facts that i_0 is unique, $\mathcal{O}_2^0$ is self-adjoint and $(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^* = \mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1}$. $\square$

Let us compute the term that appears in (1.4.3):

$$\mathrm{III}_{a,k} := I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^{j}I_{2j}. \tag{1.4.5}$$

With (1.3.21), we know that

$$P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^{j}I_{2j} = \frac{1}{(4\pi)^j}\frac{1}{2^j j!}\mathscr{P}\left(\mathscr{R}_{x_0}^j\right)^* I_{2j}, \tag{1.4.6}$$

and

$$\begin{aligned} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}&(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N \\ &= \frac{1}{(4\pi)^k}\frac{1}{2^k k!}I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}(\mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P})I_0. \end{aligned} \tag{1.4.7}$$

Let

$$\begin{aligned} R_{k\bar{m}\ell\bar{q}} &= \left\langle R^{TX}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar{z}_m}\right)\frac{\partial}{\partial z_\ell},\frac{\partial}{\partial \bar{z}_q}\right\rangle_{x_0}, \\ R^E_{k\bar{\ell}} &= R^E_{x_0}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar{z}_\ell}\right). \end{aligned} \tag{1.4.8}$$

By [49, Lemma 3.1], we know that

$$R_{k\bar{m}\ell\bar{q}} = R_{\ell\bar{m}k\bar{q}} = R_{k\bar{q}\ell\bar{m}} = R_{\ell\bar{q}k\bar{m}} \text{ and } r^X_{x_0} = 8R_{m\bar{m}q\bar{q}}. \tag{1.4.9}$$

Once again, our $\mathcal{O}_2^0$ correspond to the $\mathcal{O}_2$ of [49] (see (1.2.33) and [49, (3.13),(3.16b)]), so we can use [49, (4.6)] to get:

$$\mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P} = \left(\frac{1}{6}b_m b_q R_{k\bar{m}\ell\bar{q}}z_k z_\ell + \frac{4}{3}b_q R_{\ell\bar{k}k\bar{q}}z_\ell - \frac{\pi}{3}b_q R_{k\bar{m}\ell\bar{q}}z_k z_\ell \bar{z}'_m + b_q R^E_{\ell\bar{q}}z_\ell\right)\mathscr{R}_{x_0}^k\mathscr{P}, \tag{1.4.10}$$

Set

$$\begin{aligned} a &= \frac{1}{6}b_m b_q R_{k\bar{m}\ell\bar{q}}z_k z_\ell, & b &= \frac{4}{3}b_q R_{\ell\bar{k}k\bar{q}}z_\ell, \\ c &= -\frac{\pi}{3}b_q R_{k\bar{m}\ell\bar{q}}z_k z_\ell \bar{z}'_m, & d &= b_q R^E_{\ell\bar{q}}z_\ell. \end{aligned} \tag{1.4.11}$$

Thanks to (1.2.45), (1.2.46) and (1.4.10), we find

$$\mathscr{L}_0{}^{-1}\mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P}I_0 = \left(\frac{a}{4\pi(2+2k)} + \frac{b+c+d}{4\pi(1+2k)}\right)\mathscr{R}_{x_0}^k\mathscr{P}I_0, \tag{1.4.12}$$

and by induction, (1.4.7) becomes

$$\begin{aligned} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}&(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N \\ &= \frac{1}{(4\pi)^{j+1}}\frac{1}{2^k k!}I_{2j}\mathscr{R}_{x_0}^{j-k}\left(\frac{a}{(2+2k)\cdots(2+2j)} + \frac{b+c+d}{(1+2k)\cdots(1+2j)}\right)\mathscr{R}_{x_0}^k\mathscr{P}I_0. \end{aligned} \tag{1.4.13}$$

Lemma 1.4.4. *We have:*

$$\begin{aligned} (a\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) &= \frac{1}{6}r^X_{x_0}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z), & (b\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) &= -\frac{1}{3}r^X_{x_0}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z), \\ (c\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) &= 0, & (d\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) &= -2R^E_{q\bar{q}}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z). \end{aligned} \tag{1.4.14}$$

Proof. This lemma is a consequence of the relations (1.2.44) and (1.4.9). For instance, we will compute $(b\mathscr{R}^k_{x_0}\mathscr{P})(0,Z)$, the other terms are similar.

$$\begin{aligned}(b\mathscr{R}^k_{x_0}\mathscr{P})(0,Z) &= \left(\frac{4}{3}b_q R_{\ell\bar{k}k\bar{q}} z_\ell \mathscr{R}^k_{x_0}\mathscr{P}\right)(0,Z)\\ &= \frac{4}{3}R_{\ell\bar{k}k\bar{q}}\mathscr{R}^k_{x_0}\left((z_\ell b_q - 2\delta_{\ell q})\,\mathscr{P}\right)(0,Z)\\ &= -\frac{8}{3}R_{\ell\bar{k}k\bar{\ell}}\mathscr{R}^k_{x_0}\mathscr{P}(0,Z) = -\frac{1}{3}r^X_{x_0}\mathscr{R}^k_{x_0}\mathscr{P}(0,Z).\end{aligned}$$

□

Using (1.3.22), (1.4.5), (1.4.6) and (1.4.12), we find

$$\mathrm{III}_{a,k}(0,0)=I_{2j}\mathrm{C}_j(j)\mathscr{R}^{j-k}_{x_0}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r^X_{x_0}-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}R^E_{q\bar{q}}\right]\mathscr{R}^k_{x_0}\left(\mathscr{R}^j_{x_0}\right)^* I_{2j}. \tag{1.4.15}$$

Notice that $2R^E_{q\bar{q}} = R^E_{x_0}\left(\sqrt{2}\frac{\partial}{\partial z_q}, \sqrt{2}\frac{\partial}{\partial \bar{z}_q}\right) = R^E_{x_0}(w_q, \overline{w}_q) = \sqrt{-1}R^E_{\Lambda,x_0}$ by definition. Consequently,

$$\mathrm{III}_a(0,0) =$$

$$I_{2j}\mathrm{C}_j(j)\sum_{k=0}^{j}\mathscr{R}^{j-k}_{x_0}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r^X_{x_0}-\frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x_0}\right]\mathscr{R}^k_{x_0}\left(\mathscr{R}^j_{x_0}\right)^* I_{2j}. \tag{1.4.16}$$

1.4.3 The two other terms

In this subsection, we suppose that $j \geq 1$ (cf. Remark 1.4.1). Moreover, the existence of any $\mathscr{L}_0{}^{-1}$ appearing in this section follows from the reasoning done in Remark 1.4.3, and this operator will be used without further precision.

Due to (1.2.30), we have

$$\mathcal{O}_3^{+2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^{+2}_{E_0}(tZ)\right)|_{t=0} = z_i\frac{\partial \mathscr{R}.}{\partial z_i}(0) + \bar{z}_i\frac{\partial \mathscr{R}.}{\partial \bar{z}_i}(0) \text{ and} \tag{1.4.17}$$

$$\mathcal{O}_4^{+2} = \frac{z_i z_j}{2}\frac{\partial^2 \mathscr{R}.}{\partial z_i \partial z_j}(0) + z_i\bar{z}_j\frac{\partial^2 \mathscr{R}.}{\partial z_i \partial \bar{z}_j}(0) + \frac{\bar{z}_i\bar{z}_j}{2}\frac{\partial^2 \mathscr{R}.}{\partial \bar{z}_i \partial \bar{z}_j}(0). \tag{1.4.18}$$

The sum I can be decomposed inyo 3 'sub-sums': I_a, I_b and I_c in which the two $\mathcal{O}_3$'s appearing are respectively both at the left of P^N, either side of P^N or both at the right of P^N (see Remark 1.4.1). As usual, we have $\mathrm{I}_c = (\mathrm{I}_a)^*$.

In the same way, we can decompose $\mathrm{II} = \mathrm{II}_a + \mathrm{II}_b$: in II_a the $\mathcal{O}_4$ appears at the left of P^N, and in II_b at the right of P^N. Once again, $\mathrm{II}_b = (\mathrm{II}_a)^*$.

Computation of $\mathrm{I}_b(0,0)$

To compute I_b, we first compute the value at $(0,Z)$ of the kernel of

$$A_k := I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k\mathscr{P}I_0. \tag{1.4.19}$$

By (1.3.21) and (1.4.17),

$$\begin{aligned}A_k &= I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\mathscr{R}^k_{x_0}\mathscr{P}I_0\\ &= \frac{1}{(4\pi)^k}\frac{1}{2^k k!}I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\left[z_i\frac{\partial \mathscr{R}.}{\partial z_i}(0) + \bar{z}_i\frac{\partial \mathscr{R}.}{\partial \bar{z}_i}(0)\right]\mathscr{R}^k_{x_0}\mathscr{P}I_0.\end{aligned} \tag{1.4.20}$$

Now by Theorem 1.2.10, if $s \in N$, then $z_i s \in N$, so by the same calculation as in (1.3.21),

$$\begin{aligned}
&\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0^{-1}\left[z_i\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\right]\mathscr{R}_{x_0}^k\mathscr{P}I_0\right)(0,Z)\\
&=\frac{1}{(4\pi)^j}\frac{1}{2^j j!}\left(I_{2j}\left[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\right]z_i\mathscr{P}I_0\right)(0,Z)=0. \qquad (1.4.21)
\end{aligned}$$

Now by (1.2.43) and the formula (1.2.48), we have

$$(b_i^+\mathscr{P})(Z,Z')=0 \text{ and } (b_i\mathscr{P})(Z,Z')=2\pi(\bar{z}_i-\bar{z}_i')\mathscr{P}(Z,Z'). \qquad (1.4.22)$$

Thus,

$$\begin{aligned}
&\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0^{-1}\left[\bar{z}_i\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\right]\mathscr{R}_{x_0}^k\mathscr{P}I_0\right)(Z,Z') \qquad (1.4.23)\\
&=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0^{-1}\left[\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\left(\frac{b_i}{2\pi}+\bar{z}_i'\right)\mathscr{P}I_0\right)(Z,Z')\\
&=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-1}\left[\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\right.\\
&\qquad\qquad\left.\times\left(\frac{1}{4\pi(2k+2+1)}\frac{b_i}{2\pi}+\frac{1}{4\pi(2k+2)}\bar{z}_i'\right)\mathscr{P}I_0\right)(Z,Z')\\
&=\frac{1}{(4\pi)^j}\frac{1}{2^j j!}\left(I_{2j}\left[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\bar{z}_i'\mathscr{P}I_0\right)(Z,Z')\\
&+\frac{1}{(4\pi)^j}\frac{1}{2^k k!\prod_{k+1}^j(2\ell+1)}\left(I_{2j}\left[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\frac{b_i}{2\pi}\mathscr{P}I_0\right)(Z,Z').
\end{aligned}$$

For the last two lines, we used that if $s \in N$, then $\mathscr{L}(b_i s) = 4\pi b_i s$ (see Theorem 1.2.10). Thus, by (1.1.12) and (1.4.20)–(1.4.23)

$$\begin{aligned}
A_k(0,Z)&=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0^{-1}\left[\bar{z}_i\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\right]\mathscr{R}_{x_0}^k\mathscr{P}I_0\right)(0,Z)\\
&=\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)I_{2j}\left[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\bar{z}_i\mathscr{P}(0,Z)I_0. \qquad (1.4.24)
\end{aligned}$$

We know that $(\bar{z}_i\mathscr{P})^*=z_i\mathscr{P}$, and $\int_{\mathbb{C}^n} z_m\bar{z}_q e^{-\pi|z|^2}dZ=\frac{1}{\pi}\delta_{mq}$, so

$$\begin{aligned}
(A_{k_1}A_{k_2}{}^*)(0,0)&=\frac{1}{\pi}I_{2j}\left[\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k_1)\Big)\mathscr{R}_{x_0}^{j-k_1-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^{k_1}\right]\\
&\qquad\times\left[\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k_2)\Big)\mathscr{R}_{x_0}^{j-k_2-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^{k_2}\right]^*I_{2j}. \qquad (1.4.25)
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{I}_b(0,0)&=\frac{1}{\pi}I_{2j}\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]\\
&\qquad\times\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\right]^*I_{2j}. \qquad (1.4.26)
\end{aligned}$$

Computation of $\mathrm{I}_a(0,0)$ and $\mathrm{I}_c(0,0)$

First recall that $\mathrm{I}_c(0,0) = (\mathrm{I}_a(0,0))^*$, so we just need to compute $\mathrm{I}_a(0,0)$. By the definition of $\mathrm{I}_a(0,0)$, for it to be non-zero, it is necessary to have $j \geq 2$, which will be assumed in this paragraph. Let

$$A_{k,\ell} := I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-k-\ell-2}(\mathscr{L}_0^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^k(\mathscr{L}_0^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0, \tag{1.4.27}$$

the sum $\mathrm{I}_a(0,0)$ is then given by

$$\mathrm{I}_a(0,0) = \int_{\mathbb{R}^{2n}} \left(\sum_{k,\ell} A_{k,\ell}(0,Z)\right) \times \left(\frac{1}{(4\pi)^j}\frac{1}{2^j j!} I_{2j}\mathscr{R}_{x_0}^j \mathscr{P} I_0\right)^* (Z,0) dv_{TX}(Z). \tag{1.4.28}$$

In the following, we will set

$$\tilde{b}_i := \frac{b_i}{2\pi}. \tag{1.4.29}$$

Using the same method as in (1.2.44), (1.4.21), (1.4.22) and (1.4.23), we find that there exist constants $C^1_{k,\ell}$, $C^2_{k,\ell}$ given by

$$\begin{aligned} C^1_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}} \frac{1}{2^{k+\ell+1}(k+\ell+1)!}, \\ C^2_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}} \frac{1}{2^\ell \ell! \prod_{\ell+1}^{k+\ell+1}(2s+1)}, \end{aligned} \tag{1.4.30}$$

such that

$$\begin{aligned} &(\mathscr{L}_0^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^k(\mathscr{L}_0^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0 \qquad (1.4.31)\\ &= \mathscr{L}_0^{-1}\Big\{ \frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_i z_{i'} + \frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_{i'}(\tilde{b}_i + \bar{z}'_i) \\ &\qquad + \frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0} z_i (C^2_{k,\ell}\tilde{b}_{i'} + C^1_{k,\ell}\bar{z}'_{i'}) \\ &\qquad\qquad + \frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0}(\tilde{b}_i + \bar{z}'_i)(C^2_{k,\ell}\tilde{b}_{i'} + C^1_{k,\ell}\bar{z}'_{i'})\Big\} \mathscr{P} I_0 \\ &= \mathscr{L}_0^{-1}\Big\{ \frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_i z_{i'} + \frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell}\Big(\tilde{b}_i z_{i'} + \frac{\delta_{ii'}}{\pi} + z_{i'}\bar{z}'_i\Big) \\ &\qquad + \frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0}\Big(C^2_{k,\ell}\Big(\tilde{b}_{i'} z_i + \frac{\delta_{ii'}}{\pi}\Big) + C^1_{k,\ell} z_i \bar{z}'_{i'}\Big) \\ &\qquad\qquad + \frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0}\Big(C^2_{k,\ell}(\tilde{b}_i\tilde{b}_{i'} + \bar{z}'_i\tilde{b}_{i'}) + C^1_{k,\ell}(\tilde{b}_i\bar{z}'_{i'} + \bar{z}'_i\bar{z}'_{i'})\Big)\Big\} \mathscr{P} I_0, \end{aligned}$$

Using Theorem 1.2.10, (1.2.44) and (1.4.22), we see that there exist constants $C^i_{j,k,\ell}$, $i = 3, \ldots, 10$, such that

$$\begin{aligned} C^3_{j,k,\ell} &= C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^j(2s)}, \quad C^4_{j,k,\ell} = C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^j(2s+1)}, \\ C^5_{j,k,\ell} &= C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^j(2s+1)}, \quad C^6_{j,k,\ell} = C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^j(2s)}, \end{aligned} \tag{1.4.32}$$

and

$$\begin{aligned}
A_{k,\ell}(0,Z) = I_{2j}\Big(&\mathscr{R}_{x_0}^{j-k-\ell-2}\Big\{\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\Big(C^3_{j,k,\ell}\frac{\delta_{ii'}}{\pi}+C^4_{j,k,\ell}\tilde{b}_i z_{i'}\Big) \qquad (1.4.33)\\
&+\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_{i'}}(0)\Big(C^5_{j,k,\ell}\tilde{b}_{i'}z_i+\frac{\delta_{ii'}}{\pi}C^6_{j,k,\ell}\Big)\\
&+\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_{i'}}(0)\Big(C^7_{j,k,\ell}\tilde{b}_i\tilde{b}_{i'}+C^8_{j,k,\ell}\bar{z}_i\tilde{b}_{i'}+C^9_{j,k,\ell}\tilde{b}_i\bar{z}_{i'}+C^{10}_{j,k,\ell}\bar{z}_i\bar{z}_{i'}\Big)\Big\}\mathscr{R}^\ell_{x_0}\mathscr{P}I_0\Big)(0,Z)\\
= I_{2j}\mathscr{R}_{x_0}^{j-k-\ell-2}&\Big\{\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\frac{C^3_{j,k,\ell}-C^4_{j,k,\ell}}{\pi}+\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\frac{C^6_{j,k,\ell}-C^5_{j,k,\ell}}{\pi}\\
&+\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_{i'}}(0)\Big(4C^7_{j,k,\ell}-2C^8_{j,k,\ell}-2C^9_{j,k,\ell}+C^{10}_{j,k,\ell}\Big)\bar{z}_i\bar{z}_{i'}\Big\}\mathscr{R}^\ell_{x_0}\mathscr{P}(0,Z)I_0.
\end{aligned}$$

Now with $\int\bar{z}_i\bar{z}_{i'}\mathscr{P}(0,Z)\mathscr{P}(Z,0)dZ=0$, we can rewrite (1.4.28):

$$\begin{aligned}
\mathrm{I}_a(0,0)=\frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{k,\ell}\mathscr{R}_{x_0}^{j-k-\ell-2}&\Big\{(C^3_{j,k,\ell}-C^4_{j,k,\ell})\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\\
&+(C^6_{j,k,\ell}-C^5_{j,k,\ell})\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\Big\}\mathscr{R}^\ell_{x_0}\left(\mathscr{R}^j_{x_0}\right)^*I_{2j}. \qquad (1.4.34)
\end{aligned}$$

By (1.1.12), (1.4.30) and (1.4.32),

$$\begin{aligned}
C^3_{j,k,\ell}&=\mathrm{C}_j(j), & C^4_{j,k,\ell}&=\mathrm{C}_j(k+\ell+1),\\
C^5_{j,k,\ell}&=\mathrm{C}_j(\ell), & C^6_{j,k,\ell}&=\mathrm{C}_j(\ell)\prod_{s=k+\ell+2}^{j}\Big(1+\frac{1}{2s}\Big).
\end{aligned} \qquad (1.4.35)$$

We can now write (1.4.34) more precisely:

$$\begin{aligned}
\mathrm{I}_a(0,0)=\frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{q=0}^{j-2}\sum_{m=0}^{q}&\Big\{\Big(\mathrm{C}_j(j)-\mathrm{C}_j(q+1)\Big)\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^m_{x_0}\\
&+\mathrm{C}_j\,(m)\left[\prod_{q+2}^{j}\Big(1+\frac{1}{2s}\Big)-1\right]\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}^m_{x_0}\Big\}\left(\mathscr{R}^j_{x_0}\right)^*I_{2j}.
\end{aligned} \qquad (1.4.36)$$

Computation of $\mathrm{II}(0,0)$

Recall that $\mathrm{II}(0,0)=\mathrm{II}_a(0,0)+(\mathrm{II}_a(0,0))^*$. The computation of $\mathrm{II}_a(0,0)$ is very similar to the computation of $\mathrm{I}_a(0,0)$, and is simpler, so we will follow the same method.

Let

$$B_k:=I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k\mathscr{P}I_0,$$

the sum $\mathrm{II}_a(0,0)$ is then given by

$$\mathrm{II}_a(0,0)=\int_{\mathbb{R}^{2n}}\Big(\sum_k B_k(0,Z)\Big)\times\Big(\frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}^j_{x_0}\mathscr{P}I_0\Big)^*(Z,0)dv_{TX}(Z). \qquad (1.4.37)$$

Using (1.4.18), we can repeat what we have done for (1.4.31) and (1.4.33). We find that there is a constant C (which we do not need to compute) such that

$$B_k(0,Z) = I_{2j}\left\{\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\mathrm{C}_j(j)-\mathrm{C}_j(k)}{\pi}\right.$$
$$\left.+\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial\bar{z}_i\partial\bar{z}_{i'}}(0)\mathscr{R}_{x_0}^{k}C\frac{\bar{z}_i\bar{z}_{i'}}{2}\right\}\mathscr{P}(0,Z)I_0. \quad (1.4.38)$$

Thus, we get

$$\mathrm{II}_a(0,0) = \frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\left(\mathscr{R}_{x_0}^{j}\right)^{*}I_{2j}. \quad (1.4.39)$$

Conclusion

In order to conclude the proof of Theorem 1.1.7, we just have to put the pieces together. But before that, as we want to write the formulas in a more intrinsic way, we have to note that since we trivialized $\Lambda^{0,\bullet}(T^*X)\otimes E$ with $\nabla^{\Lambda^{0,\bullet}\otimes E}$, since $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial\bar{z}_i}$, and thanks to [49, (5.44),(5.45)], we have

$$\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0) = \frac{1}{\sqrt{2}}\left(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}_{\cdot}\right)(x_0), \quad \frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0) = \frac{1}{\sqrt{2}}\left(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i}\mathscr{R}_{\cdot}\right)(x_0) \text{ and}$$
$$\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0) = -\frac{1}{4}(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x_0).$$

With these remarks and equations (1.4.3), (1.4.16), (1.4.26), (1.4.36), (1.4.39) used in decomposition (1.4.2), we get Theorem 1.1.7.

1.5 The third coefficient in the asymptotic expansion when the first two vanish

In this section, we prove Theorem 1.1.8. Using (1.3.6), we know that

$$I_{2j}\boldsymbol{b}_{2j+2}I_{2j}(0,0) = I_{2j}\mathscr{F}_{4j+4}I_{2j}(0,0). \quad (1.5.1)$$

Here again, we will first decompose this term into several terms in Section 1.5.1, and then in Sections 1.5.2, 1.5.3 and 1.5.4 we handle them separately.

For the rest of the section we fix $j\in[\![1,n]\!]$, and we suppose that

$$I_{2j}\boldsymbol{b}_{2j}I_{2j}(0,0) = I_{2j}\boldsymbol{b}_{2j+1}I_{2j}(0,0) = 0. \quad (1.5.2)$$

By Theorems 1.1.4 and 1.1.7, this is equivalent to

$$\begin{cases}\mathscr{R}_x^j = 0\\ \mathcal{T}_0(j) = 0.\end{cases} \quad (1.5.3)$$

For every smoothing operator F acting on $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ that appears in this section, we will denote by $F(Z,Z')$ its smooth kernel with respect to $dv_{TX}(Z')$. Moreover, recall that every operators A we have

$$\mathrm{Pos}[A] = AA^* \quad\text{and}\quad \mathrm{Sym}[A] = A+A^*. \quad (1.5.4)$$

1.5.1 Decomposition of the computation

With the same reasoning as in Section 1.4.1, we see that in the decomposition (1.3.11) of $I_{2j}\mathscr{F}_{4j+4}I_{2j}$, the non-zero terms $\int_\delta T^\eta_{\mathbf{r}}(\lambda)d\lambda$ appearing satisfy $k = 2j$, $2j+1$ or $2j+2$. Moreover, we can find the possible terms by adding one term to or modifying the subscript of the terms we mentioned in section 1.4.1. The list of possible terms is as follows.

I. The terms such that $k = 2j+2$.

   Here, there are up to three indices i such that $\eta_i = N$ is and are in $\{j, j+1, j+2\}$. Moreover, the only $\mathscr{O}_\ell$'s appearing are some $\mathscr{O}_2$'s. The possibilities are now

   I-a) $2j+2$ times $\mathscr{O}_2^{\pm 2}$,

   I-b) $2j$ times $\mathscr{O}_2^{\pm 2}$ and 2 times $\mathscr{O}_2^0$.

II. The terms such that $k = 2j+1$.

   Here, there are one or two indices i such that $\eta_i = N$ is and are in $\{j, j+1\}$, and there is exactly one $\mathscr{O}_\ell^0$ that appears in these terms. We regroup them in relation to the $\mathscr{O}_{r_i}$ that they contain.

   II-a) $2j$ times $\mathscr{O}_2^{\pm 2}$ and 1 time $\mathscr{O}_4^0$,

   II-b) $2j-1$ times $\mathscr{O}_2^{\pm 2}$, 1 time $\mathscr{O}_2^0$ and 1 time $\mathscr{O}_4^{\pm 2}$,

   II-c) $2j-1$ times $\mathscr{O}_2^{\pm 2}$, 1 time $\mathscr{O}_3^{\pm 2}$ and 1 time $\mathscr{O}_3^0$,

   II-d) $2j-2$ times $\mathscr{O}_2^{\pm 2}$, 1 time $\mathscr{O}_2^0$ and 2 times $\mathscr{O}_3^{\pm 2}$.

III. The terms such that $k = 2j$.

   Here, the i_0 such that $\eta_{i_0} = N$ is unique and equal to j, and no $\mathscr{O}_\ell^0$ appears in these terms. We regroup them in relation to the $\mathscr{O}_{r_i}$ that they contain.

   III-a) $2j-4$ times $\mathscr{O}_2^{\pm 2}$ and 4 times $\mathscr{O}_3^{\pm 2}$,

   III-b) $2j-3$ times $\mathscr{O}_2^{\pm 2}$, 2 times $\mathscr{O}_3^{\pm 2}$ and 1 time $\mathscr{O}_4^{\pm 2}$,

   III-c) $2j-2$ times $\mathscr{O}_2^{\pm 2}$ and 2 times $\mathscr{O}_4^{\pm 2}$,

   III-d) $2j-2$ times $\mathscr{O}_2^{\pm 2}$, 1 time $\mathscr{O}_3^{\pm 2}$ and 1 time $\mathscr{O}_5^{\pm 2}$

   III-e) $2j-1$ times $\mathscr{O}_2^{\pm 2}$ and 1 time $\mathscr{O}_6^{\pm 2}$.

This list seem quite long, but fortunately most of the terms will ultimately vanish due to the fact that they make appear some terms involved in $I_{2j}\boldsymbol{b}_{2j}I_{2j}$ and $I_{2j}\boldsymbol{b}_{2j+1}I_{2j}$.

In the sequel, the contribution to the third coefficient of the terms of type I-a), I-b), etc. will be denoted by $T_{\text{I-a)}}$, $T_{\text{I-b)}}$, etc.

1.5.2 The terms of type I

We first begin with the following observation.

Lemma 1.5.1. *For any j-tuple $(a_1, \dots, a_j)$ of positive integers, we have*

$$X_{(a_1,\dots,a_j)} := I_{2j}\left(\prod_{i=1}^{j} \mathscr{L}_0{}^{-a_i}\mathscr{O}_2^{+2}\right)P^N = 0. \tag{1.5.5}$$

Proof. This is an easy extension of the computation (1.3.21), using the fact that $\mathscr{R}_x^j = 0$. ∎

The terms of type I-a)

In these terms, only some $\mathcal{O}_2^{\pm 2}$ appears, so there is either a unique i_0 such that $\eta_{i_0} = N$ which is then equal to j or $j+2$, either exactly two such i_0's which are then j and $j+2$.

Each term that is in the second case is a sum of term of the form

$$-X_{(a_1,\dots,a_j)}\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-b}\mathcal{O}_2^{+2}X^*_{(a'_1,\dots,a'_j)} \tag{1.5.6}$$

with $a_i, a'_k, b \in \{1,2\}$ (exactly one is equal to 2). By Lemma 1.5.1, these terms vanish.

Now, each term in the first case is equal or adjoint to a term of the form

$$I_{2j}\left(\prod_{i=1}^{j+2}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{\varepsilon_i}\right)\mathscr{P}I_0\left(I_{2j}\left(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2}\right)^j\mathscr{P}I_0\right)^*, \tag{1.5.7}$$

where $\varepsilon_i \in \{-2,+2\}$ (exactly one of the ε_i's is equal to -2). By Lemma, 1.5.1, these terms vanish.

Finally, every term of type I-a) vanishes and $T_{\text{I-a)}} = 0$.

The terms of type I-b)

Using Lemma 1.5.1 as above, we see that the only non-zero terms of this type satisfy that before the first index i such that $\eta_i = N$ and after the last, there must be a $\mathcal{O}_2^0$ appearing. As a consequence, the cases where two or three η_i's are equal to N lead to vanishing terms. We now deal with the terms where $\eta_{j+1} = N$ and for $i \neq j+1$, $\eta_i = N^\perp$. Such terms are of the form

$$\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k\mathscr{P}\Big)\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k'}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k'}\mathscr{P}\Big)^*, \tag{1.5.8}$$

for $0 \le k, k' \le j$. By the computations of Section 1.4.2, and in particular (1.4.15), we find

$$\begin{aligned} & I_{2j}\Big((\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k\Big)\mathscr{P}I_0 = \\ & \quad I_{2j}\mathscr{R}_x^{j-k}\left[\frac{1}{6}\Big(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\Big)r_x^X - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x}\right]\mathscr{R}_x^k\mathscr{P}I_0. \end{aligned} \tag{1.5.9}$$

Observe that r^X commutes with $\mathscr{R}$, and that $\mathscr{R}^j = 0$, so the contribution of the terms of type I-b) is finally $T_{\text{I-b)}} = \mathrm{Pos}\,[\mathcal{T}_3'(j)]$.

1.5.3 The terms of type II

The terms of type II-a)

In these terms, there are either only $\mathcal{O}_2^{-2}$ appearing at the right of the first P^N or only $\mathcal{O}_2^{+2}$ appearing at the left of the last P^N. Either way, all these terms vanish by Lemma 1.5.1. Hence $T_{\text{II-a)}} = 0$.

The terms of type II-b)

For these terms, there are two possibilities.

Firstly, there are two indices i such that $\eta_i = N$, and then they are equal to j and $j+1$. In this case, either before the first P^N, either after the last, there are j $\mathcal{O}_2^{+2}$'s (or $\mathcal{O}_2^{-2}$'s) that appear, so all these terms vanish.

Secondly, there is a unique i_0 such that $\eta_{i_0} = N$ and it is equal to j or $j+1$. We denote by S_1 (resp. S_2) the sum of the term for which $i_0 = j$ (resp. $i_0 = j+1$). Then $S_1 = S_2^*$ and

$$\begin{aligned} S_2 = \sum_{k,\ell} &\left\{ I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k \mathscr{P} I_0 \right\} \\ &\qquad\qquad \times \left\{ I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0 \right\}^* \\ = &\left\{ \sum_k I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k \mathscr{P} I_0 \right\} \\ &\qquad\qquad \times \left\{ \sum_\ell I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0 \right\}^* . \end{aligned} \tag{1.5.10}$$

By (1.4.15) and (1.4.39) we find that the contribution of the terms of type II-b), i.e. $S_1(0,0) + S_2(0,0)$, is $T_{\text{II-b)}} = \mathrm{Sym}\,[\mathcal{T}_2(j)\mathcal{T}_3'(j)^*]$.

The terms of type II-c)

The computation is the same as for terms of type II-b), except that in the case of a unique i_0 such that $\eta_{i_0} = N$, in (1.5.10) we must replace $\mathcal{O}_4^{+2}$ by $\mathcal{O}_3^{+2}$ and $\mathcal{O}_2^0$ by $\mathcal{O}_3^0$. Recall that A_k has been defined in (1.4.19). By (1.4.24) and (1.5.3), we find that the contribution of the terms of type II-c) is the symmetric operator associated to

$$\left\{ \sum_k A_k \right\} \left\{ \sum_\ell I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0{}^{-1}\mathcal{O}_3^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0 \right\}^* . \tag{1.5.11}$$

By (1.5.3) we get $T_{\text{II-c)}} = 0$.

The terms of type II-d)

Here again, we have the same possibilities concerning the indices i such that $\eta_i = N$ as for terms of types II-b) or II-c). If there are two such indices, then they are equal to j and $j+1$ and between the two corresponding P^N's we will have the term $\mathcal{O}_2^0$. By (1.4.4), these terms vanish.

We now suppose that there is a unique i_0 such that $\eta_{i_0} = N$. Then $i_0 = j$ or $j+1$. As $\mathscr{R}_x^j = 0$, any term where the two $\mathcal{O}_3$'s and the $\mathcal{O}_2^0$ appear on the same side of P^N will vanish. A term where there is 1 $\mathcal{O}_3$ at the left and 1 $\mathcal{O}_3$ at the right of P^N is equal or adjoint to

$$I_{2j} \left(\prod_{i=1}^{j+1} \mathscr{L}_0{}^{-1}\mathcal{O}_{a_i}^{\varepsilon_i} \right) \mathscr{P} I_0 \times A_k^*, \tag{1.5.12}$$

where $a_i = 2$ or 3 and $\varepsilon_i = +2$ except for exactly one i_1 satisfying $a_{i_1} = 2$ (for which $\varepsilon_{i_1} = 0$). By (1.4.24) and (1.5.3), the sum of this terms vanishes.

Finally, the only possibility is that the two $\mathcal{O}_3$'s appear on the same side of P^N, and $\mathcal{O}_2^0$ on the other side. Recall that $A_{k,\ell}$ has been define in (1.4.27). The sum of the remaining terms is equal to

$$\mathrm{Sym} \left[\left\{ \sum_{k,\ell} A_{k,\ell} \right\} \left\{ \sum_m I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-m}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^m \mathscr{P} I_0 \right\}^* \right] . \tag{1.5.13}$$

As a consequence, the contribution of terms of type II-d) is $T_{\text{II-d)}} = \mathrm{Sym}\,[\mathcal{T}_1(j)\mathcal{T}_3'(j)^*]$.

1.5.4 The terms of type III

The computations rely on similar arguments that the ones undertaken in Sections 1.5.2 and 1.5.3. We will therefore give the contribution of each sub-type directly.

The terms of type III-a)

The contribution of these terms is $T_{\text{III-a)}} = \operatorname{Pos}[\mathcal{T}_1(j)]$.

The terms of type III-b)

The contribution of these terms is $T_{\text{III-b)}} = \operatorname{Sym}[\mathcal{T}_1(j)\mathcal{T}_2(j)^*]$

The terms of type III-c)

The contribution of these terms is $T_{\text{III-c)}} = \operatorname{Pos}[\mathcal{T}_2(j)]$.

The terms of type III-d)

The sum of all these terms vanishes: $T_{\text{III-d)}} = 0$.

The terms of type III-e)

This terms vanishes, so that $T_{\text{III-e)}} = 0$.

By all the computations of Sections 1.5.2, 1.5.3 and 1.5.4, we get Theorem 1.1.8.

Chapter 2

G-invariant Holomorphic Morse inequalities

2.1 Introduction

Morse Theory investigates the topological information carried by Morse functions on a manifold and in particular their critical points. Let f be a Morse function on a compact manifold of real dimension n. We suppose that f has isolated critical points. Let m_j, $(0 \leq j \leq n)$ be the the number of critical points of f of Morse index j, and let b_j be the Betti numbers of the manifold. Then the strong Morse inequalities states that for $0 \leq q \leq n$,

$$\sum_{j=0}^{q}(-1)^{q-j}b_j \leq \sum_{j=0}^{q}(-1)^{q-j}m_j, \tag{2.1.1}$$

with equality if $q = n$. From (2.1.1), we get the weak Morse inequalities:

$$b_j \leq m_j \qquad \text{for} \quad 0 \leq j \leq n. \tag{2.1.2}$$

In his seminal paper [65], Witten gave an analytic proof of the Morse inequalities by analyzing the spectrum of the Schrödinger operator $\Delta_t = \Delta + t^2|df|^2 + tV$, where $t > 0$ is a real parameter and V an operator of order 0. For $t \to +\infty$, Witten shows that the spectrum of Δ_t approaches in some sense the spectrum of a sum of harmonic oscillators attached to the critical point of f.

In [27], Demailly established analogous asymptotic Morse inequalities for the Dolbeault complex associated with high tensor powers $L^p := L^{\otimes p}$ of a holomorphic Hermitian line bundle (L, h^L) over a compact complex manifold (M, J). The inequalities of Demailly give asymptotic bounds on the Morse sums of the Betti numbers of $\bar{\partial}$ on L^p in terms of certain integrals of the Chern curvature R^L of (L, h^L). More precisely, we define $\dot{R}^L \in \operatorname{End}(T^{(1,0)}M)$ by $g^{TM}(\dot{R}^L u, \overline{v}) = R^L(u, \overline{v})$ for $u, v \in T^{(1,0)}M$, where g^{TM} is a J-invariant Riemannian metric on TM. We denote by $M(\leq q)$ the set of points where $\dot{R}^L$ is non-degenerate and have at most q negative eigenvalues, and we set $n = \dim_{\mathbb{C}} M$. Then we have for $0 \leq q \leq n$

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M, L^p) \leq \frac{p^n}{n!}\int_{M(\leq q)}(-1)^q\left(\frac{\sqrt{-1}}{2\pi}R^L\right)^n + o(p^n), \tag{2.1.3}$$

with equality if $q = n$. Here $H^j(M, L^p)$ denotes the Dolbeault cohomology in bidegree $(0, j)$, which is also the j-th group of cohomology of the sheaf of holomorphic sections of L^p.

These inequalities have found numerous applications. In particular, Demailly used them in [27] to find new geometric characterizations of Moishezon spaces, which improve Siu's solution in [56, 57] of the Grauert-Riemenschneider conjecture [36]. Another notable application of the holomorphic Morse inequalities is the proof of the effective Matsusaka theorem by Siu [58, 29]. Recently, Demailly used these inequalities in [30] to prove a significant step of a generalized version of the Green-Griffiths-Lang conjecture.

To prove these inequalities, the key remark of Demailly was that in the formula for the Kodaira Laplacian $\square_p$ associated with L^p, the metric of L plays formally the role of the Morse function in the paper Witten [65], and that the parameter p plays the role of the parameter t. Then the Hessian of the Morse function becomes the curvature of the bundle. The proof of Demailly was based on the study of the semi-classical behavior as $p \to +\infty$ of the spectral counting functions of $\square_p$. Subsequently, Bismut reproved in [6] the holomorphic Morse inequalities by adapting his heat kernel proof of the Morse inequality [5]. The key point is that we can compare the left hand side of (2.1.3) with the alternate trace of the heat kernel acting on forms of degree $\leq q$, i.e.,

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M,L^p) \leq \sum_{j=0}^{q}(-1)^{q-j}\,\mathrm{Tr}^{\Omega^{0,j}(M,L^p)}\Big[\exp\Big(-\frac{u}{p}\square_p\Big)\Big], \tag{2.1.4}$$

with equality if $q = n$. Then, Bismut obtained the holomorphic Morse inequalities by showing the convergence of the heat kernel thanks to probability theory. Demailly [28] and Bouche [20] gave an analytic approach of this result. In [46], Ma and Marinescu gave a new of this convergence, replacing the probabilistic arguments of Bismut [6] by arguments inspired by the analytic localization techniques of Bismut-Lebeau [14, Chap. 11].

When the bundle L is positive, (2.1.3) is a consequence of the Hirzebruch-Riemann-Roch theorem and of the Kodaira vanishing theorem, and reduces to

$$\dim H^0(M,L^p) = \frac{p^n}{n!}\int_M \Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(p^n). \tag{2.1.5}$$

In this case, a local estimate can be obtained by the study of the asymptotic of the Bergman kernel (the kernel of the orthogonal projection from $\mathscr{C}^\infty(M,L^p)$ onto $H^0(M,L^p)$) when $p \to +\infty$. We refer to [46] and the reference therein for the study of the Bergman kernel.

In the equivariant case, a connected compact Lie group G acts on M and its action lifts on L. When L is positive, Ma and Zhang [51] have studied the invariant Bergman kernel, i.e., the kernel of the projection from $\mathscr{C}^\infty(M,L^p)$ onto the G-invariant part of $H^0(M,L^p)$. Let μ be the moment map associated with the G-action on M (see (2.1.7)). Ma and Zhang [51] established that the invariant Bergman kernel concentrate to any neighborhood U of $\mu^{-1}(0)$, and that near $\mu^{-1}(0)$, we have a full off-diagonal asymptotic development. They also obtain a fast decay of the invariant Bergman kernel in the normal directions to $\mu^{-1}(0)$, which does not appear in the classical case.

In this chapter, we establish G-invariant holomorphic Morse inequalities in under certain natural condition in the context of Ma-Zhang [51] but without the assumption that L is positive.

More precisely, we consider an action of a connected compact Lie group G on a compact complex manifold M and two G-equivariant vector bundles L and E on M, with L of rank 1, and we establish asymptotic holomorphic Morse inequalities similar to (2.1.3) for the G-invariant part of the Dolbeault cohomology of $L^p \otimes E$ (see Theorems 2.1.3 and 2.1.5). To do so, we define a "moment map" $\mu\colon M \to \mathrm{Lie}(G)$ by the Kostant formula and we define the reduction of M under natural hypothesis on $\mu^{-1}(0)$ (see Assumption 2.1.1).

Our inequalities are then given in term of the curvature of the bundle induced by L on this reduction, and the integral in (2.1.3) will be over subsets of the reduction.

A new feature in our setting when compared to Demailly's result is the localization near $\mu^{-1}(0)$. We use a heat kernel method inspired by [6] (see also [46, Sect. 1.6-1.7]), the key behind that an analogue of (2.1.4) still holds (see Lemma 2.1.7) for the Kodaira Laplacian restricted to the space of invariant forms. We show that the heat kernel will concentrate in any neighborhood U of $\mu^{-1}(0)$, and we study the asymptotic of the heat kernel near $\mu^{-1}(0)$. For this last part, we work with the operator induced by the Kodaira Laplacian on the quotient of U. However, as we will have to integrate the heat kernel in the normal directions to $\mu^{-1}(0)$, we need a more precise convergence result that in [46, Sect. 1.6]. Indeed we also need to prove a uniform fast decay of the heat kernel in the normal directions, which is analogous to the decay encountered in [51, Thm. 0.2]. Our approach is largely inspired by [51].

Note that in the literature, there exists another type of holomorphic Morse inequalities [66, 53, 67], which relate the Dolbeault cohomology groups of the fixed point-set of a compact Kähler manifold M endowed with an action of a compact connected Lie group G to the Dolbeault cohomology groups of M itself.

We now give more details about our results. Let (M, J) be a connected compact complex manifold. Let $n = \dim_{\mathbb{C}} M$. Let (L, h^L) be a holomorphic Hermitian line bundle on M, and (E, h^E) a Hermitian complex vector bundle on M. We denote the Chern (i.e., holomorphic and Hermitian) connections of L and E respectively by ∇^L and ∇^E, and their respective curvatures by $R^L = (\nabla^L)^2$ and $R^E = (\nabla^E)^2$. Let ω be the first Chern form of (L, h^L), i.e., the $(1,1)$-form defined by

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2.1.6}$$

We *do not assume* that ω is a positive $(1,1)$-form.

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Let $d = \dim_{\mathbb{R}} G$. We assume that G acts holomorphically on (M, J), and that the action lifts in a holomorphic action on L and E. We assume that h^L and h^E are preserved by the G-action. Then R^L, R^E and ω are G-invariant forms.

In the sequel, if F is any G-representation, we denote by F^G the space of elements of F invariant under the action of G. The infinitesimal action of $K \in \mathfrak{g}$ on any F will be denoted by $\mathcal{L}_K^F$, or simply by $\mathcal{L}_K$ when it entails no confusion.

For $K \in \mathfrak{g}$, let K^M be the vector field on M induced by K (see (2.2.2)). We can define a map $\mu\colon M \to \mathfrak{g}^*$ by the Kostant formula

$$\mu(K) = \frac{1}{2i\pi}\left(\nabla^L_{K^M} - \mathcal{L}_K\right). \tag{2.1.7}$$

Then for any $K \in \mathfrak{g}$, $d\mu(K) = i_{K^M}\omega$. Moreover the set defined by

$$P = \mu^{-1}(0) \tag{2.1.8}$$

is stable by G.

We make the following assumption:

Assumption 2.1.1. 0 *is a regular value of* μ.

Under Assumption 2.1.1, P is a submanifold. Moreover, by Lemma 2.3.2, G acts locally freely on P, so that the quotient $M_G = P/G$ is an orbifold, which we call the *reduction* of M. For definition and basic properties of orbifolds, we refer to [46, Sect. 5.4] for instance.

We denote by TY the tangent bundle of the orbits in P. As G acts locally freely on P, we know that $TY = \mathrm{Span}(K^M, K \in \mathfrak{g})$ and that it is a vector bundle on P.

The following analogue of the classical Kähler reduction (see [37]) holds.

Theorem 2.1.2. *The complex structure J on M induces a complex structure J_G on M_G, for which the orbifold bundles L_G, E_G on M_G induced by L, E are holomorphic. Moreover, the form ω descends to a form ω_G on M_G and if R^{L_G} is the Chern curvature of L_G for the metric h^{L_G} induced by h^L, then*

$$\omega_G = \frac{\sqrt{-1}}{2\pi} R^{L_G}. \tag{2.1.9}$$

Finally, π_* induces an isomorphism

$$\ker \omega_G \simeq (\ker \omega)|_P. \tag{2.1.10}$$

Let b^L be the bilinear form on TM

$$b^L(\cdot\,,\cdot) = \frac{\sqrt{-1}}{2\pi} R^L(\cdot\,, J\cdot) = \omega(\cdot\,, J\cdot). \tag{2.1.11}$$

Then we will show in Lemma 2.3.3 that when restricted to $TY \times TY$, the bilinear form b^L is non-degenerate on P. In particular, the signature of $b^L|_{TY\times TY}$ is constant on P. We denote by $(r, d-r)$ this signature, i.e., in any orthogonal (with respect to b^L) basis of $TY|_P$, the matrix of b^L will have r negative diagonal elements and $d-r$ positive diagonal elements.

We define $\dot{R}^{L_G} \in \mathrm{End}(T^{(1,0)}M_G)$ by $g(\dot{R}^{L_G}u, \overline{v}) = R^{L_G}(u, \overline{v})$ for $u, v \in T^{(1,0)}M_G$, where g is a J_G-invariant Riemannian metric on the orbifold tangent bundle TM_G. We denote by $M_G(q)$ the set of $x \in M_G$ such that $\dot{R}^{L_G}_x$ is invertible and has exactly q negative eigenvalues, with the convention that if $q \notin \{0, \dots, n-d\}$, then $M_G(q) = \emptyset$. Set $M_G(\leq q) = \cup_{i=0}^q M_G(i)$. Note that $M_G(q)$ does not depend on the metric g.

As G preserves every structure we are given, it acts naturally on the Dolbeault cohomology $H^\bullet(M, L^p \otimes E)$. The following theorem is the main result of this chapter.

Theorem 2.1.3. *Assume that G acts effectively on M (i.e., the only element of G acting as Id_M is the identity). Then as $p \to +\infty$, the following strong Morse inequalities hold for $q \in \{1, \dots, n\}$*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \leq \mathrm{rk}(E) \frac{p^{n-d}}{(n-d)!} \int_{M_G(\leq q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}), \tag{2.1.12}$$

with equality for $q = n$.

In particular, we get the weak Morse inequalities

$$\dim H^q(M, L^p \otimes E)^G \leq \mathrm{rk}(E) \frac{p^{n-d}}{(n-d)!} \int_{M_G(q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}). \tag{2.1.13}$$

Remark 2.1.4. We assume temporarily that G acts freely on P, so that M_G is a manifold.

If L is positive, then ω is a Kähler form and μ is a genuine moment map. Moreover, (M_G, ω_G) is the usual Kähler reduction of M (see [37]). By [62, Thm. 0.2], quantization and reduction commute in this case: for p large enough,

$$H^\bullet(M, L^p \otimes E)^G \simeq H^\bullet(M_G, L_G^p \otimes E_G). \tag{2.1.14}$$

In particular, as in the non equivariant case, Theorem 2.1.3 is, in this case, a consequence of (2.1.14) and of the Hirzebruch-Riemann-Roch theorem and of the Kodaira vanishing theorem, both applied in this case on M_G.

We prove here that even if ω is degenerate or if G does not act freely on P, under Assumption 2.1.1, we have the same estimate for $\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M, L^p\otimes E)^G$ as the one given by the holomorphic Morse inequalities on M_G for $\sum_{j=0}^{q-r}(-1)^{q-j}\dim H^j(M, L^p\otimes E)^G$.

Theorem 2.1.3 is in fact a particular case of the more general Theorem 2.1.5 below.

Set

$$G^0 = \{g\in G \,:\, g\cdot x = x \text{ for any } x\in M\}, \tag{2.1.15}$$

which is a finite normal subgroup of G. Note that we will see in (2.6.23) that we also have $G^0 = \{g\in G \,:\, g\cdot x = x \text{ for any } x\in P\}$.

Observe that $\dim(L^p_v\otimes E_v)^{G^0}$ does not depend on $v\in M$. We will thus denote it simply by $\dim(L^p\otimes E)^{G^0}$.

Theorem 2.1.5. *As $p\to+\infty$, the following strong Morse inequalities hold for $q\in\{1,\dots,n\}$*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M, L^p\otimes E)^G \\ \leq \dim(L^p\otimes E)^{G^0}\frac{p^{n-d}}{(n-d)!}\int_{M_G(\leq q-r)}(-1)^{q-r}\omega_G^{n-d} + o(p^{n-d}), \tag{2.1.16}$$

with equality for $q=n$.

In particular, we get the weak Morse inequalities

$$\dim H^q(M, L^p\otimes E)^G \leq \dim(L^p\otimes E)^{G^0}\frac{p^{n-d}}{(n-d)!}\int_{M_G(q-r)}(-1)^{q-r}\omega_G^{n-d} + o(p^{n-d}). \tag{2.1.17}$$

Remark 2.1.6. The integer $\dim(L^p\otimes E)^{G^0}$ depends on p. However, as G^0 is finite and acts by rotations on L, there exists $k\in\mathbb{N}$ (a divisor of the cardinal of G^0) such that G^0 acts trivially on L^k. In particular, we have $\dim(L^{kp}\otimes E)^{G^0} = \dim E^{G^0}$.

We now explain what are the main steps of our proof.

Let g^{TM} be a J- and G-invariant metric on TM. Let dv_M be the corresponding Riemannian volume on M, and let ∇^{TM} be the Levi-Civita connection on (TM, g^{TM}). Let $\bar\partial^{L^p\otimes E}$ be the Dolbeault operator acting on $\Omega^{0,\bullet}(M, L^p\otimes E)$. Let $\bar\partial^{L^p\otimes E,*}$ be its dual with respect to the L^2 product induced by g^{TM}, h^L and h^E. We set

$$D_p = \sqrt{2}\left(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\right), \tag{2.1.18}$$

and we denote by $e^{-uD_p^2}$ the associated heat kernel.

We denote P_G the orthogonal projection from $\Omega^{0,\bullet}(M, L^p\otimes E)$ onto $\Omega^{0,\bullet}(M, L^p\otimes E)^G$. Let $(P_G e^{-\frac{u}{p}D_p^2}P_G)(v,v')$ be the Schwartz kernel of $P_G e^{-\frac{u}{p}D_p^2}P_G$ with respect to $dv_M(v')$.

Note that the operator D_p^2 acts on $\Omega^{0,\bullet}(M, L^p\otimes E)^G$ (i.e., commutes with P_G) and preserves the $\mathbb{Z}$-grading. we denote by $\mathrm{Tr}_q[P_G e^{-\frac{u}{p}D_p^2}P_G]$ the trace of $P_G e^{-\frac{u}{p}D_p^2}P_G$ acting on $\Omega^{0,q}(M, L^p\otimes E)$. We then have an analogue of (2.1.4):

Theorem 2.1.7. *For any $u > 0$, $p \in \mathbb{N}^*$ and $0 \leq q \leq n$, we have*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \leq \sum_{j=0}^{q} (-1)^{q-j} \operatorname{Tr}_j \left[P_G e^{-\frac{u}{p} D_p^2} P_G\right], \tag{2.1.19}$$

with equality for $q = n$.

We now give the estimates on $P_G e^{-\frac{u}{p} D_p^2} P_G$ to treat the right-hand side of (2.1.19).

Let U be a small open G-invariant neighborhood of P, such that G acts locally freely on its closure $\overline{U}$.

First, we have away from P the following theorem. The analogous result of Ma-Zhang for the Bergman kernel is [51, Thm. 0.1].

Theorem 2.1.8. *For any fixed $u > 0$ and $k, \ell \in \mathbb{N}$, there exists $C > 0$ such that for any $p \in \mathbb{N}^*$ and $v, v' \in M$ with $v, v' \in M \setminus U$,*

$$\left| P_G e^{-\frac{u}{p} D_p^2} P_G(v, v') \right|_{\mathscr{C}^\ell} \leq C p^{-k}, \tag{2.1.20}$$

where $|\cdot|_{\mathscr{C}^\ell}$ is the $\mathscr{C}^\ell$-norm induced by ∇^L, ∇^E, ∇^{TM}, h^L, h^E and g^{TM}.

We now turn to the "near P" asymptotic of the heat kernel. To explain simply this asymptotic, we assume now that G acts freely on P. We can thus also assume that G acts freely on $\overline{U}$. Let $B = U/G$. Then M_G and B are here genuine manifolds. We will explain in Section 2.6.2 how to adapt the proof of Theorems 2.1.3 and 2.1.5 to the case of a locally free action.

We again denote by $TY = \operatorname{Span}(K^M, K \in \mathfrak{g})$ the tangent bundle of the orbits in U. By Lemma 2.3.3, we have

$$TU = TY \oplus (TY)^{\perp_{b^L}}. \tag{2.1.21}$$

Then we can choose the horizontal bundles of the fibrations $U \to B$ and $P \to M_G$ to be

$$T^H U = (TY)^{\perp_{b^L}} \quad \text{and} \quad T^H P = T^H U\Big|_P \cap TP. \tag{2.1.22}$$

Indeed, using (2.1.21) and the fact that $TY \subset TP$, we see that

$$TP = TY \oplus T^H P. \tag{2.1.23}$$

Let $g^{T^H P}$ be a G-invariant and J-invariant metric on $T^H P$. Let g^{TY} be a G-invariant metric on TY and let g^{JTY} be the G-invariant metric on JTY induced by J and g^{TY}. Then by (2.3.19), we can chose the metric g^{TM} on TM so that on P:

$$g^{TM}|_P = g^{TY}|_P \oplus g^{JTY}|_P \oplus g^{T^H P}. \tag{2.1.24}$$

We will use this condition on g^{TM} in the rest of the introduction as well as in Sections 2.5.1-2.6.2.

Suppose that U is small enough so that it can be identified with a ε-neighborhood, $\varepsilon > 0$, of the zero section of the normal bundle N of P in U via exponential map. We denote the corresponding coordinate by $v = (y, Z^\perp) \in U$ with $y \in P$ and $Z^\perp \in N_y$. Note that by (2.1.23) and (2.1.24) we can identify N_y and JTY_y.

Let $\mathbf{J} \in \operatorname{End}(TM|_P)$ be such that on P

$$\omega = g^{TM}(\mathbf{J}\cdot, \cdot). \tag{2.1.25}$$

We also denote by $\mathbf{J}$ the induced operator on B.

By (2.5.3), the normal bundle N_G of M_G in B can be identified with the bundle $(JTY)_B$ induced on B by JTY (see Section 2.2). In particular, if $\pi(y) = x$, we keep the same notation for an element of N_y and the corresponding element in $N_{G,x}$. Then we will see in Section 2.5.3 that $\mathbf{J}$ stabilize the bundle N_G, and we will define $a_i^\perp \in \mathbb{R}^*$ such that

$$\operatorname{Sp}(\mathbf{J}^2|_{N_G}) = -\frac{1}{4\pi^2}\{a_1^{\perp,2}, \dots, a_d^{\perp,2}\}. \tag{2.1.26}$$

Let g^{TB} be the metric on TB induced by g^{TM} and T^HP. Let g^{N_G} be the induced metric on N_G and dv_{N_G} the corresponding volume form. For $x \in M_G$, let $\{e_i^\perp\}_{i=1}^d$ be an orthonormal basis of $N_{G,x}$ such that $\mathbf{J}_x^2 e_i^\perp = -\frac{1}{4\pi^2} a_i^\perp(x) e_i^\perp$. We can then identify $\mathbb{R}^d$ with $N_{G,x}$ via the map

$$(Z_1^\perp, \dots, Z_d^\perp) \in \mathbb{R}^d \mapsto Z^\perp = \sum_{i=1}^d Z_i^\perp e_i^\perp. \tag{2.1.27}$$

We now define the operator $\mathscr{L}_x^\perp$ acting on $N_{G,x} \simeq \mathbb{R}^d$ by

$$\mathscr{L}_x^\perp = -\sum_{i=1}^d \Big((\nabla_{e_i^\perp})^2 - |a_i^\perp Z_i^\perp|^2 \Big) - \sum_{j=1}^d a_j^\perp \tag{2.1.28}$$

where ∇_U denotes the ordinary differentiation operator on $\mathbb{R}^d$ in the direction U. We denote by $e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z'^\perp)$ the heat kernel of $\mathscr{L}_x^\perp$ with respect to $dv_{N_{G,x}}(Z'^\perp)$. Note that we have an explicit formula for $e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z'^\perp)$ (see (2.6.12)), but we do not give it to have a simpler asymptotic formula for the heat kernel.

Let g^{TM_G} be the metric on M_G induced by g^{TM} and T^HP and dv_{M_G} the corresponding volume form. We denote by $\langle \cdot\,, \cdot \rangle_G$ the $\mathbb{C}$-bilinear extension of g^{TM_G} on $TM_G \otimes \mathbb{C}$. Then we can identify R^{L_G} with the Hermitian matrix $\dot{R}^{L_G} \in \operatorname{End}(T^{(1,0)}M_G)$ such that for $V, V' \in T^{(1,0)}M_G$,

$$R^{L_G}(V, V') = \langle \dot{R}^{L_G} V, \overline{V'} \rangle_G. \tag{2.1.29}$$

Let also $\{w_j\}$ be a local orthonormal frame of $T^{(1,0)}M_G$ with dual frame $\{w^j\}$. Set

$$\omega_{G,d} = -\sum_{i,j} R^{L_G}(w_i, \overline{w}_j) \overline{w}^j \wedge i_{\overline{w}_i}. \tag{2.1.30}$$

Let h be the G-invariant function on M given by

$$h(x) = \sqrt{\operatorname{vol}(G.x)}, \tag{2.1.31}$$

and let $\kappa \in \mathscr{C}^\infty(TB|_{M_G})$ be the function defined by $\kappa|_{M_G} = 1$ and for $x \in M_G$, $Z \in T_xB$,

$$dv_B(x, Z) = \kappa(x, Z) dv_{T_{x_0}B}(Z) = \kappa(x, Z) dv_{M_G}(x) dv_{N_{G,x}}(Z). \tag{2.1.32}$$

The following result is a version of [51, Thm. 2.21] for the heat kernel in our situation.

Theorem 2.1.9. *Assume that G acts freely on P. For any fixed $u > 0$ and $m \in \mathbb{N}$, we have the following convergence as $p \to +\infty$ for $|Z^\perp| < \varepsilon$:*

$$\begin{aligned} h(y, Z^\perp)^2 \big(P_G e^{-\frac{u}{p} D_p^2} P_G\big)\big((y, Z^\perp), (y, Z^\perp)\big) = \\ \frac{\kappa^{-1}(x, Z^\perp)}{(2\pi)^{n-d}} \frac{\det(\dot{R}_x^{L_G}) e^{2u\omega_d(x)}}{\det\big(1 - \exp(-2u\dot{R}_x^{L_G})\big)} e^{-u\mathscr{L}_x^\perp}(\sqrt{p}Z^\perp, \sqrt{p}Z^\perp) \otimes \operatorname{Id}_E p^{n-d/2} \\ + O\big(p^{n-d/2-1/2}(1 + \sqrt{p}|Z^\perp|)^{-m}\big), \end{aligned} \tag{2.1.33}$$

where $x = \pi(y) \in M_G$ and the term $O(\cdot)$ is uniform. The convergence is in the $\mathscr{C}^\infty$-topology in $y \in P$. Here, we use the convention that if an eigenvalue of $\dot{R}^{L_G}_{x_0}$ is zero, then its contribution to $\frac{\det(\dot{R}^{L_G}_{x_0})}{\det\left(1-\exp(-u\dot{R}^{L_G}_{x_0})\right)}$ is $\frac{1}{2u}$.

From Theorems 2.1.7, 2.1.8 and 2.1.9, we get Theorem 2.1.3 in the case where G acts freely on P by integration on M.

This chapter is organized as follows. In Section 2.2, we recall some constructions associated with a principal bundle. In Section 2.3, we apply the constructions and results of Section 2.2 to our situation to define the reduction of M and to descend the different objects we are given on it, thus proving Theorem 2.1.2. In Section 2.4 we prove the localization of the heat kernel near P, i.e., Theorem 2.1.8. In Sections 2.5, we assume for simplicity that G acts freely on P and $\overline{U}$, and study the asymptotic of the heat kernel near P by localizing the problem and studying a rescaled Laplacian on B. We thus obtain Theorem 2.1.9. Finally, in Section 2.6, we prove the G-invariant holomorphic Morse inequalities (Theorems 2.1.3 and 2.1.5) and we show how to use Theorem 2.1.5 to get estimates on the other isotypic components of the cohomology $H^\bullet(M, L^p \otimes E)$.

2.2 Connections and Laplacians associated to a principal bundle

In this section, we review some results of [51, Chp. 1] for the convenience of the reader.

Let G be a connected compact Lie group of dimension d that acts smoothly and locally freely on the left on a smooth manifold M of dimension m. Then $\pi\colon M \to B = M/G$ is a G-principal bundle and B is an orbifold. We denote by TY the relative tangent bundle of this fibration.

Note that in [51, Chp. 1], Ma and Zhang assumed that G acts freely on M, but as explained in the introduction of [51, Chp. 1] and in [51, Sect. 4.1], the results of [51, Chp. 1] extend to the case where G acts only locally freely, essentially because when we work on orbifold quotients, we in fact work with invariant sections on M.

Let g^{TM} be a G-invariant metric on TM, and ∇^{TM} the corresponding Levi-Civita connection on TM. We denote by T^HM the orthogonal complement of TY in TM. For $U \in TB$, we denote by U^H the horizontal lift of U in TM, that is $\pi_* U^H = U$ and $U^H \in T^HM$. Let $\theta\colon TM \to \mathfrak{g}$ be the connection form corresponding to T^HM, and let Θ be its curvature, i.e., the horizontal form such that

$$\Theta(U^H, V^H) = -P^{TY}[U^H, V^H], \tag{2.2.1}$$

where P^{TY} is the natural projection $TM = TY \oplus T^HM \to TY$.

The metric g^{TM} induces a metric g^{TY} (resp. g^{T^HM}) on TY (resp. T^HM). Let g^{TB} be the metric on TB induced by g^{T^HM}, and let ∇^{TB} be the corresponding Levi-Civita connection.

Let (F, h^F) be a G-equivariant Hermitian vector bundle with G-equivariant Hermitian connection ∇^F. Then G acts on $\mathscr{C}^\infty(M, F)$ by $(g.s)(x) = g.s(g^{-1}x)$.

Any $K \in \mathfrak{g}$ induces a vector field K^M on M given by

$$K^M_x = \left.\frac{\partial}{\partial s}\right|_{s=0} e^{-sK}.x. \tag{2.2.2}$$

For $K \in \mathfrak{g}$, recall that $\mathcal{L}_K$ is the infinitesimal action of K on any G representation. Let $\mu^F \in \mathscr{C}^\infty(M, \mathfrak{g}^* \otimes \mathrm{End}(F))$ be defined by

$$\mu^F(K) = \nabla^F_{K^M} - \mathcal{L}_K. \tag{2.2.3}$$

Using the identification $TY \simeq M\times\mathfrak{g}$, we can identify μ^F with $\widetilde{\mu}^F \in \mathscr{C}^\infty(M, TY\otimes \mathrm{End}(F))^G$ such that

$$\langle \widetilde{\mu}^F, K^M\rangle = \mu^F(K). \tag{2.2.4}$$

Let F_B be the orbifold bundle on B induced by F, i.e., $F_{B,x} = \mathscr{C}^\infty(\pi^{-1}(x), F|_{\pi^{-1}(x)})^G$. Then there is a canonical isomorphism

$$\pi_G \colon \mathscr{C}^\infty(M,F)^G \xrightarrow{\sim} \mathscr{C}^\infty(B,F_B). \tag{2.2.5}$$

The invariant metric h^F induces a metric h^{F_B} on F_B. For $s \in \mathscr{C}^\infty(B,F_B)$ and $U\in TB$, we define

$$\nabla^{F_B}_U s := \nabla^F_{U^H} s. \tag{2.2.6}$$

Observe that ∇^{F_B} is the restriction of the connection $\nabla^F - \mu(\theta)$ to $\mathscr{C}^\infty(M,F)^G$. Let R^{F_B} be the curvature of ∇^{F_B}. Then by [51, (1.18)] we have for $V, V' \in TB$

$$R^{F_B}(V,V') = R^F(V^H, V'^H) - \mu^F(\Theta)(V,V'). \tag{2.2.7}$$

Let dv_M be the Riemannian volume on (M, g^{TM}). We endow $\mathscr{C}^\infty(M,F)$ with the L^2 product induced by g^{TM} and h^F:

$$\langle s, s'\rangle = \int_M \langle s, s'\rangle_{h^F}(x) dv_M(x). \tag{2.2.8}$$

In the same way, g^{TB} and h^{F_B} induce a L^2 product $\langle\cdot\,,\cdot\rangle$ on $\mathscr{C}^\infty(B,F_B)$.

For $x\in M$, we denote by $\mathrm{vol}(G.x)$ the volume of the orbit of $G.x$ endowed with the restriction of g^{TM}. Define the G-invariant function h on M by

$$h(x) = \sqrt{\mathrm{vol}(G.x)}. \tag{2.2.9}$$

Then h define a function on B, which is still denoted by h. Note that h is smooth only on the regular part of B. However, we can extend it continuously to get a smooth function $\widehat{h}$ on B. Then $\widehat{h}$ also define a smooth function on U.

The map

$$\Phi := \widehat{h}\pi_G \colon (\mathscr{C}^\infty(M,F)^G, \langle\cdot\,,\cdot\rangle) \to (\mathscr{C}^\infty(B,F_B), \langle\cdot\,,\cdot\rangle) \tag{2.2.10}$$

is then an isometry.

Let $\{u_i\}_{i=1}^m$ be an orthonormal frame of TM. For any Hermitian bundle with Hermitian connection (E, h^E, ∇^E) on M, the Bochner Laplacians Δ^E, Δ_M are given by

$$\Delta^E = -\sum_{i=1}^m \left((\nabla^E_{u_i})^2 - \nabla^E_{\nabla^{TM}_{u_i} u_i} \right), \quad \Delta_M = \Delta^{\mathbb{C}}. \tag{2.2.11}$$

Let $\{f_l\}_{l=1}^d$be a G-invariant orthonormal frame of TY with dual frame $\{f^l\}_{l=1}^d$, and let $\{e_i\}_{i=1}^{m-d}$ be an orthonormal frame of TB. Then $\{e_i^H, f_l\}$ form an orthonormal frame of TM.

For $\sigma, \sigma' \in TY\otimes \mathrm{End}(F)$, let $\langle \sigma,\sigma'\rangle_{g^{TY}} \in \mathrm{End}(F)$ be the contraction of the part of $\sigma\otimes\sigma'$ in $TY\otimes TY$ with g^{TY}. Note that

$$\langle \widetilde{\mu}^F, \widetilde{\mu}^F\rangle_{g^{TY}} = \sum_{l=1}^d \langle \widetilde{\mu}^F, f_l\rangle^2_{g^{TY}} \in \mathrm{End}(F). \tag{2.2.12}$$

Theorem 2.2.1. *As an operator on $\mathscr{C}^\infty(B,F_B)$, $\Phi\Delta^F\Phi^{-1}$ is given by*

$$\Phi\Delta^F\Phi^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F\rangle_{g^{TY}} - \widehat{h}^{-1}\Delta_B \widehat{h}. \tag{2.2.13}$$

Proof. This is proved in [51, Thm. 1.3]. □

2.3 The reduction of M and the Laplacian on B

This Section is organized as follows. In Section 2.3.1 we apply the constructions and results of Section 2.2 to our situation to define the reduction of M and to descend the different objects we are given on it. We prove, under Assumption 2.1.1, some properties of the reduction that are well-known in the case where ω is positive and get Theorem 2.1.2. In Section 2.3.2, we compute the operator induced on U/G by the Kodaira Laplacian.

We use here the notations of the introduction. In particular, let (M, J) be a connected compact complex manifold of dimension n, let (L, h^L) be a holomorphic Hermitian line bundle on M and (E, h^E) a Hermitian complex vector bundle on M. We denote the associated Chern curvatures by R^L and R^E. Let $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ be the first Chern form of (L, h^L), which is not assumed to be positive. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Let $d = \dim_{\mathbb{R}} G$. We assume that G acts holomorphically on (M, J), and that the action lifts in a holomorphic action on L and E. We assume that h^L and h^E are preserved by the G-action.

Recall that ∇^L denotes the Chern connection of (L, h^L) and that the moment map μ is defined by $2i\pi\mu(K) = \nabla^L_{K^M} - \mathcal{L}_K$ for $K \in \mathfrak{g}$. Let $P = \mu^{-1}(0)$ and U a small tubular neighborhood of P. Finally, we set $M_G = P/G$.

2.3.1 The reduction of M

We begin by proving the following result.

Lemma 2.3.1. *The map μ is smooth on M and is linear in K. Moreover, it is moment map of the G-action on M, i.e., μ is G-invariant and for any $K \in \mathfrak{g}$,*

$$d\mu(K) = i_{K^M}\omega. \tag{2.3.1}$$

Proof. First, as both $\nabla^L_{K^M}$ and $\mathcal{L}_K$ satisfies the Leibniz rules and preserves h^L, we know that $\nabla^L_{K^M} - \mathcal{L}_K$ is $\mathscr{C}^\infty(M)$-linear, thus, under the canonical isomorphism $\mathrm{End}(L) = \mathbb{C}$,

$$\nabla^L_{K^M} - \mathcal{L}_K \in \mathscr{C}^\infty(M, i\mathbb{R}). \tag{2.3.2}$$

This proves the first part of Lemma 2.3.1.

As ∇^L is G-invariant, we have $g \cdot (\nabla^L_Y s) = \nabla^L_{g_* Y}(g \cdot s)$ for $Y \in \mathscr{C}^\infty(M, TM)$, $s \in \mathscr{C}^\infty(M, L)$ and $g \in G$. Thus, taking $g = e^{-tK}$ for $K \in \mathfrak{g}$ and differentiating at $t = 0$, we get

$$\mathcal{L}_K \nabla^L_Y s = \nabla^L_{[K^M, Y]} s + \nabla^L_Y \mathcal{L}_K s, \qquad \text{that is} \quad [\mathcal{L}_K, \nabla^L] = 0. \tag{2.3.3}$$

Using the definition of μ (2.1.7), (2.3.3) becomes

$$(2i\pi\mu(K) + \nabla^L_{K^M})\nabla^L_Y s = \nabla^L_{[K^M, Y]} s + \nabla^L_Y (2i\pi\mu(K) + \nabla^L_{K^M}) s. \tag{2.3.4}$$

This, together with (2.1.6), yields to

$$Y(\mu(K)) = \omega(K^M, Y), \tag{2.3.5}$$

which is (2.3.1)

Finally, it is easy to prove that $g_* K^M = (\mathrm{Ad}_g K)^M$ and $g \cdot (\mathcal{L}_K s) = \mathcal{L}_{\mathrm{Ad}_g K}(g \cdot s)$, so

$$2i\pi\, g \cdot (\mu(K) s) = \big(\nabla^L_{(\mathrm{Ad}_g K)^M} - \mathcal{L}_{\mathrm{Ad}_g K}\big)(g \cdot s), \tag{2.3.6}$$

and thus

$$\mu(g^{-1}x) = \mathrm{Ad}^*_{g^{-1}}\mu. \tag{2.3.7}$$

The proof of Lemma 2.3.1 is complete. $\square$

Lemma 2.3.2. *The group G acts locally freely on P.*

Proof. By (2.3.1), we have for $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$,

$$\omega(K^M, V)_x = \big(d_x\mu(V)\big)(K). \tag{2.3.8}$$

In particular, if $K^M = 0$, then $\big(d_x\mu(V)\big)(K)$ vanishes for all $V \in T_xM$. Assumption 2.1.1, the differential $d_x\mu\colon T_xM \to \mathfrak{g}^*$ is surjective, hence $K = 0$. □

Lemma 2.3.3. *When restricted to $TY \times TY$, the bilinear form b^L is non-degenerate on P.*

Proof. First, observe that for $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$, equations (2.1.11) and (2.3.1) yield

$$b^L(K^M, JV)_x = -\omega(K^M, V)_x = -\big(d_x\mu(V)\big)(K). \tag{2.3.9}$$

Let $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$. Then by (2.3.9)

$$JV \in (TY)^{\perp_{b^L}}\Big|_P \iff d_x\mu(V) = 0 \iff V \in TP, \tag{2.3.10}$$

the last equivalence coming from the fact that $P = \mu^{-1}(0)$. In particular, $\dim (TY)^{\perp_{b^L}} = \dim TP = 2n - d$, the last identity coming from the fact that 0 is a regular value of μ. Moreover, $\dim TY = d$ (because G acts locally freely on U) and $TY + (TY)^{\perp_{b^L}} = TU$. This is possible only if this sum is direct, i.e., $TY \cap (TY)^{\perp_{b^L}} = \{0\}$. We have proved our lemma. □

By Lemma 2.3.3, we have

$$TU = TY \oplus (TY)^{\perp_{b^L}}. \tag{2.3.11}$$

Then we can choose the horizontal bundles of the fibrations $U \to B$ and $P \to M_G$ to be

$$T^HU = (TY)^{\perp_{b^L}} \quad \text{and} \quad T^HP = T^HU\Big|_P \cap TP. \tag{2.3.12}$$

Indeed, using (2.3.11) and the fact that $TY \subset TP$, we see that

$$TP = TY \oplus T^HP. \tag{2.3.13}$$

Let $(L_G, h^{L_G}, \nabla^{L_G})$ and $(E_G, h^{E_G}, \nabla^{E_G})$ be defined from (L, h^L, ∇^L), (E, h^E, ∇^E) and T^HP as indicated in Section 2.2. We also define ω_G by

$$\omega_G(V, V') = \omega(V^H, V'^H). \tag{2.3.14}$$

Note that to $P = \mu^{-1}(0)$, (2.2.7) gives

$$R^{L_B}|_{M_G}(V, V') = R^L|_P(V^H, V'^H). \tag{2.3.15}$$

From (2.1.6), (2.3.14) and (2.3.15), we see that if R^{L_G} is the curvature of ∇^{L_G}, then

$$\omega_G = \frac{\sqrt{-1}}{2\pi} R^{L_G}. \tag{2.3.16}$$

Lemma 2.3.4. *We have*

$$\begin{aligned} T^HU\Big|_P &= JTP \\ TU|_P &= TP \oplus JTY. \end{aligned} \tag{2.3.17}$$

In the second line, the sum is orthogonal with respect to b^L.

Proof. Recall that $T^HU = (TY)^{\perp_{b^L}}$. Thus the first identity in (2.3.17) follows from (2.3.10).

Concerning the second, we have for $V \in TP$ and $K \in \mathfrak{g}$,

$$b^L(JK^M, V)_x = \omega(K^M, V)_x = \big(d_x\mu(V)\big)_x(K) = 0. \tag{2.3.18}$$

Using (2.3.18) and the facts that b^L is non-degenerate on JTY and that $\dim TU = \dim TP + \dim JTY$, we get the second identity in (2.3.17). □

Using Lemma 2.3.4 and (2.1.23), we find firstly

$$TU|_P = T^HP \oplus TY \oplus JTY, \tag{2.3.19}$$

the decomposition being orthogonal for b^L, and secondly

$$T^HP = TP \cap JTP. \tag{2.3.20}$$

In particular, T^HP is stable by J, so we can define an almost-complex structure on M_G in the following way. For $V \in TM_G$, we denote V^H its lift in T^HP, and we define the almost complex structure J_G on M_G by

$$(J_GV)^H = J(V^H). \tag{2.3.21}$$

Lemma 2.3.5. *The almost complex structure J_G is integrable, thus (M_G, J_G) is a complex manifold.*

Proof. Let $u, v \in \mathscr{C}^\infty(M_G, T^{1,0}M_G)$. Then there are $U, V \in \mathscr{C}^\infty(M_G, TM_G)$ such that

$$u = U - \sqrt{-1}J_GU\ , \quad v = V - \sqrt{-1}J_GV. \tag{2.3.22}$$

Using (2.3.21), we find

$$u^H = U^H - \sqrt{-1}JU^H\ , \quad v = V^H - \sqrt{-1}JV^H \in T^{1,0}M \cap T_{\mathbb{C}}P. \tag{2.3.23}$$

As both $T^{1,0}M$ and $T_{\mathbb{C}}P$ are integrable, we have $[u^H, v^H] \in T^{1,0}M \cap T_{\mathbb{C}}P$, i.e., there is $W \in \mathscr{C}^\infty(M, TM)$ such that

$$[u^H, v^H] = W - \sqrt{-1}JW, \tag{2.3.24}$$

and moreover $W, JW \in TP$. Thus, $W \in TP \cap JTP = T^HP$ and we can write $W = X^H$ for X a section of TM_G. Hence

$$[u, v] = \pi_*[u^H, v^H] = \pi_*(X^H - \sqrt{-1}JX^H) = X - \sqrt{-1}J_GX \in T^{1,0}M_G. \tag{2.3.25}$$

By the Newlander-Nirenberg theorem, (2.3.25) means that J_G is integrable. □

Lemma 2.3.6. *The bundles L_G and E_G are holomorphic. Moreover, ∇^{L_G} and ∇^{E_G} are the respective Chern connections on L_G and E_G.*

Proof. We first prove the result for L_G.

Observe that for $U, V \in TM_G$,

$$\omega_G(J_GU, J_GV) = \omega(JU^H, JV^H) = \omega(U^H, V^H) = \omega_G(U, V). \tag{2.3.26}$$

Hence, ω_G is a $(1,1)$-form, and so is R^{L_G} by (2.3.16). We decompose ∇^{L_G} into holomorphic part and anti-holomorphic part,

$$\nabla^{L_G} = (\nabla^{L_G})^{1,0} + (\nabla^{L_G})^{0,1}. \tag{2.3.27}$$

As R^{L_G} is $(1,1)$, we have

$$\left((\nabla^{L_G})^{0,1}\right)^2 = 0. \tag{2.3.28}$$

For $s \in \mathscr{C}^\infty(M_G, L_G)$, we define

$$\bar{\partial}^{L_G} s = (\nabla^{L_G})^{0,1} s. \tag{2.3.29}$$

Let s_0 be a local frame of L_G near $x_0 \in M_G$. Then we can write $(\nabla^{L_G})^{0,1} s_0 = \alpha s_0$ for some $(0,1)$-form α. By (2.3.28), we have

$$0 = \left((\nabla^{L_G})^{0,1}\right)^2 s_0 = (\bar{\partial}\alpha) s_0. \tag{2.3.30}$$

Thus, $\bar{\partial}\alpha = 0$. By the (local) $\bar{\partial}$-lemma, there is a function f defined near x_0 such that $\bar{\partial} f = -\alpha$. Thus,

$$\bar{\partial}^{L_G} s_0 + (\bar{\partial} f) s_0 = 0. \tag{2.3.31}$$

This shows that (2.3.29) defines a holomorphic structure on L_G, for wich $e^f s_0$ is a local holomorphic frame near x_0.

Finally, ∇^{L_G} is clearly Hermitian with respect to h^{L_G}, and is holomorphic by the definition (2.3.29), so ∇^{L_G} is indeed the Chern connection on L_G.

We now turn to E_G. Here again, it is enough to prove that R^{E_G} is a $(1,1)$-form (see for instance [38, Prop. I.3.7]). As R^E is a $(1,1)$-form, (2.2.7) shows that it is equivalent to prove that $\Theta|_{T^HP\times T^HP}$ is a $(1,1)$-form.

Let $u = U - \sqrt{-1}JU$ and $v = V - \sqrt{-1}JV$ be in $(T^HP)^{1,0}$. As U, V, JU and JV are in $T^HP = TP \cap JTP$ and TP is integrable, we have $[u,v] \in T_{\mathbb{C}}P$. Moreover, as u and v are of type $(1,0)$ and J is integrable, $[u,v]$ is also of type $(1,0)$, and thus $[u,v] = -iJ[u,v] \in JT_{\mathbb{C}}P$. In conclusion, $[u,v] \in T^H_{\mathbb{C}}P$ and by (2.2.1), $\Theta(u,v) = 0$. □

Lemma 2.3.7. *We have* $\ker\omega|_P \subset T^HP$*, and* π_* *induces an isomorphism*

$$\ker\omega_G \simeq \ker\omega|_P. \tag{2.3.32}$$

Proof. Let $V \in TU|_P$ be such that $\omega(V,\cdot) = 0$. Then we also have $b^L(V,\cdot) = 0$. Thus V is in particular in $(TY)^{\perp_{b^L}} = T^HU$. Moreover, V is also orthogonal (for b^L) to JTY, so is in T^HP by Lemma 2.3.4.

As $\omega_G(\pi_*\cdot\,,\pi_*\cdot) = \omega(\cdot\,,\cdot)$, we know that π_* maps $\ker\omega|_P$ in $\ker\omega_G$, and is injective as $\ker\omega|_P \subset T^HP$. Finally, if $V \in \ker\omega_G$, then $\omega(V^H, V') = 0$ for $V' \in T^HP$. In fact, as the decomposition in (2.3.19) is orthogonal for b^L, we have $\omega(V^H, V') = 0$ for $V' \in TU|_P$, and thus $V^H \in \ker\omega$. The proof of our lemma is complete. □

By (2.3.16) and Lemmas 2.3.5, 2.3.6 and 2.3.7 we have proved Theorem 2.1.2.

2.3.2 The Kodaira Laplacian and the operator induced on B

We define the vector bundle $\mathcal{E}$, and $\mathbb{E}_p$ $(p \geq 1)$ over M by

$$\begin{aligned} \mathcal{E} &= \Lambda^{0,\bullet}(T^*M) \otimes E. \\ \mathbb{E}_p &= \Lambda^{0,\bullet}(T^*M) \otimes E \otimes L^p. \end{aligned} \tag{2.3.33}$$

Then we have $\mathbb{E}_0 = \mathcal{E}$.

Recall that g^{TM} is a J- and G-invariant metric on TM (we do not assume that (2.1.24) holds in this section). We endow $\mathscr{C}^\infty(M, \mathbb{E}_p)$ with the L^2 scalar product associated with g^{TM}, h^L and h^E as in (2.2.8). Then the Dolbeault-Dirac operator D_p defined in (2.1.18) is a formally self-adjoint operator acting on $\mathscr{C}^\infty(M, \mathbb{E}_p)$.

We now recall the Lichnerowicz formula for the Kodaira Laplacian D_p^2.

Let ∇^{TM} be the Levi-Civita connection on (M, g^{TM}). We denote by $P^{T^{(1,0)}M}$ the orthogonal projection form $TM \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}M$. Let $\nabla^{T^{(1,0)}M} = P^{T^{(1,0)}M} \nabla^{TM} P^{T^{(1,0)}M}$ be the induced connection on $T^{(1,0)}M$. We endow $\det(T^{1,0}M)$ with the metric induced by g^{TM}, and we denote by $\nabla^{\det}$ the Hermitian connection on $\det(T^{1,0}M)$ induced by $\nabla^{T^{(1,0)}M}$, and let $R^{\det}$ its curvature.

Let $(w_1, \dots, w_n)$ be an orthonormal frame of $(T^{(1,0)}M, g^{TM})$, and $(e_1, \dots, e_{2n})$ be the orthonormal frame of (TM, g^{TM}) given by

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j). \tag{2.3.34}$$

Let $\{e^k\}$ be the dual basis of $\{e_k\}$. The Clifford action of $T^*_{\mathbb{C}}M$ on $\Lambda^{0,\bullet}(T^*M)$ is defined by linearity from

$$c(w_j) := \sqrt{2}\overline{w}^j \wedge \quad \text{and} \quad c(\overline{w}_j) := -\sqrt{2} i_{\overline{w}_j}. \tag{2.3.35}$$

We define a map, still denoted by $c(\cdot)$, on $\Lambda(T^*_{\mathbb{C}}M)$ by setting for $j_1 < \cdots < j_k$:

$$c(e^{j_1} \wedge \cdots \wedge e^{j_k}) := c(e_{j_1}) \dots c(e_{j_k}). \tag{2.3.36}$$

Let Γ^{TM} and $\Gamma^{\det}$ be the connection forms of ∇^{TM} and $\nabla^{\det}$ associated to the frames $\{e_i\}$ and $w_1 \wedge \cdots \wedge w_n$. Define the the *Clifford connection* on $\Lambda^{0,\bullet}(T^*M)$ (see [46, (1.3.5)]) by the following local formula in the frame $\{\overline{w}^{i_1} \wedge \cdots \wedge \overline{w}^{i_k}\}$:

$$\nabla^{\mathrm{Cl}} = d + \frac{1}{4} \sum_{i,j} \langle \Gamma^{TM} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \Gamma^{\det}. \tag{2.3.37}$$

We also denote by ∇^{Cl} the connection on $\mathcal{E}$ induced by ∇^{Cl} and ∇^E.

Let Ω be the real $(1,1)$-form defined by

$$\Omega = g^{TM}(J\cdot, \cdot). \tag{2.3.38}$$

On $\Lambda^{0,\bullet}(T^*M)$, we define the *Bismut connection* ∇^{Bi} by

$$\nabla^{\mathrm{Bi}}_V = \nabla^{\mathrm{Cl}}_V + \frac{\sqrt{-1}}{4} c\big(i_V(\partial - \bar{\partial})\Omega\big). \tag{2.3.39}$$

This connection, along with ∇^E and ∇^L, induces connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathbb{E}_p}$ on $\mathcal{E}$ and $\mathbb{E}_p$. Moreover, we know that (see e.g. [46, Thm. 1.4.5])

$$D_p = \sum_{i=1}^{2n} c(e_i) \nabla^{\mathbb{E}_p}_{e_i}. \tag{2.3.40}$$

Let $\Delta^{\mathbb{E}_p}$ is the Bochner Laplacian on $\mathbb{E}_p$ induced by $\nabla^{\mathbb{E}_p}$. It is given by the following formula: if (g^{ij}) is the inverse of the matrix $(g_{ij}) = (g^{TM}_Z(e_i, e_j))$, then

$$\Delta^{\mathbb{E}_p} = -g^{ij} \left(\nabla^{\mathbb{E}_p}_{e_i} \nabla^{\mathbb{E}_p}_{e_j} - \nabla^{\mathbb{E}_p}_{\nabla^{TM}_{e_i} e_j} \right). \tag{2.3.41}$$

Let r^M be the scalar curvature of (M, g^{TM}). Let $\Psi_{\mathcal{E}}$ be the smooth self-adjoint section of $\mathrm{End}(\mathcal{E})$ given by

$$\Psi_{\mathcal{E}} = \frac{r^M}{4} + c(R^E + \frac{1}{2}R^{\det}) + \frac{\sqrt{-1}}{2}c(\bar{\partial}\partial\Omega) - \frac{1}{8}\big|(\partial - \bar{\partial})\Omega\big|^2. \tag{2.3.42}$$

Set also

$$\begin{aligned} \omega_d &= -\sum_{i,j} R^L(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i}, \\ \tau &= \sum_i R^L(w_i, \overline{w}_i). \end{aligned} \tag{2.3.43}$$

The Lichnerowicz formula (see for instance [46, Thm. 1.4.7 and (1.5.17)]) reads

$$D_p^2 = \Delta^{\mathbb{E}_p} - p(2\omega_d + \tau) + \Psi_{\mathcal{E}}, \tag{2.3.44}$$

Let μ^E, μ^{Bi} and $\mu^{\mathbb{E}_p}$ be the moment maps induced by ∇^E, ∇^{Bi} and $\nabla^{\mathbb{E}_p}$ as in (2.2.3). Recall that μ is defined in (2.1.7). Then we have

$$\begin{cases} \mu^L = 2i\pi\mu, \\ \mu^{\mathbb{E}_p} = 2i\pi p\mu + \mu^E + \mu^{\mathrm{Bi}}. \end{cases} \tag{2.3.45}$$

Assume now that G acts freely on P, and recall that we then choose the G-invariant neighborhood U of P so that G acts freely on its closure $\overline{U}$. Using the procedure of Section 2.2 for $U \to U/G = B$ and $g^{TM}|_U$, we can define the operator $\Phi D_p^2 \Phi^{-1}$ induced by D_p^2 on B. Thanks to Theorem 2.2.1 and (2.3.44), we find that in the case of a free G-action on P,

$$\Phi D_p^2 \Phi^{-1} = \Delta^{\mathbb{E}_{p,B}} - p(2\omega_d + \tau) + \Psi_{\mathcal{E}} - \langle \widetilde{\mu}^{\mathbb{E}_p}, \widetilde{\mu}^{\mathbb{E}_p} \rangle_{g^{TY}} - \widehat{h}^{-1}\Delta_B \widehat{h}. \tag{2.3.46}$$

Here, we have kept the same notation for an element in $\mathscr{C}^\infty(U, \mathrm{End}(\mathbb{E}_p))^G$ and the induced element in $\mathscr{C}^\infty(B, \mathrm{End}(\mathbb{E}_{p,B}))$, and we will always do in the sequel.

2.4 Localization near P

The goal of this section is to prove the localization of $P_G e^{-\frac{u}{p}D_p^2} P_G$ near P, i.e., we prove Theorem 2.1.8.

Let inj^M be the injectivity radius of (M, g^{TM}), and $\varepsilon \in]0, \mathrm{inj}^M[$.

For $x_0 \in M$, we denote by $B^M(x_0, \varepsilon)$ and $B^{T_{x_0}M}(0, \varepsilon)$ the open balls in M and $T_{x_0}M$ with center x_0 and 0 and radius ε respectively. If $\exp_{x_0}^M$ is the exponential map of M, then $Z \in B^{T_{x_0}M}(0, \varepsilon) \mapsto \exp_{x_0}^M(Z) \in B^M(x_0, \varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}M$ with $\mathbb{R}^{2n}$ via an orthonormal basis $\{e_i\}$ of $T_{x_0}M$:

$$(Z_1, \ldots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_i Z_i e_i \in T_x M. \tag{2.4.1}$$

From now on, we will always identify $B^{T_{x_0}M}(0, \varepsilon)$ and $B^M(x_0, \varepsilon)$.

Let $x_1, \ldots x_N$ be points of M such that $\{U_k = B^M(x_k, \varepsilon)\}_{k=1}^N$ is an open covering of M. On each U_k we identify E_Z, L_Z and $\Lambda^{0,\bullet}(T_Z^*M)$ to E_{x_k}, L_{x_k} and $\Lambda^{0,\bullet}(T_{x_k}^*M)$ by parallel transport with respect to ∇^E, ∇^L and ∇^{Bi} along the geodesic ray $t \in [0,1] \mapsto tZ$. We fixe for each $k = 1, \ldots, N$ an orthonormal basis $\{e_i\}_i$ of $T_{x_k}M$ (without mentioning the dependence on k).

We denote by ∇_V the ordinary differentiation operator in the direction V on $T_{x_k}M$.

Let $\{\varphi_k\}_k$ be a partition of unity subordinate to $\{U_k\}_k$. For $\ell \in \mathbb{N}$, we define a Sobolev norm $||\cdot||_{\boldsymbol{H}^\ell(p)}$ on the ℓ-th Sobolev space $\boldsymbol{H}^\ell(M, \mathbb{E}_p)$ by

$$||s||^2_{\boldsymbol{H}^\ell(p)} = \sum_k \sum_{j=0}^{\ell} \sum_{i_1,\dots,i_j=1}^{d} ||\nabla_{e_{i_1}} \dots \nabla_{e_{i_j}}(\varphi_k s)||^2_{L^2}. \tag{2.4.2}$$

The following two results are [46, Lem. 1.6.2] and [46, Prop. 1.6.4]. We reprove them here for the sake of completeness.

Lemma 2.4.1. *For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that for any $p \in N$, $u > 0$ and $s \in \boldsymbol{H}^{2m+2}(M, \mathbb{E}_p)$,*

$$||s||^2_{\boldsymbol{H}^{2m+2}(p)} \leq C_m p^{4m+4} \sum_{j=0}^{m+1} p^{-4j} ||D_p^{2j} s||_{L^2}. \tag{2.4.3}$$

Proof. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TM} along the curve $t \in [0,1] \mapsto tZ$. Then $\{\widetilde{e}_i\}_i$ is an orthonormal frame of TM.

Let Γ^E, Γ^L and Γ^{Bi} be the corresponding connection form of ∇^E, ∇^L and ∇^{Bi} with respect to any fixed frame for E, L and $\Lambda^{0,\bullet}(T^*M)$ which is parallel along the curve $t \in [0,1] \mapsto tZ$ under the trivialization on U_k. By (2.3.40), on each U_k we have

$$\nabla^{\mathbb{E}_p} = \nabla + \Gamma^{\mathrm{Bi}} + \Gamma^E + p\Gamma^L, \qquad D_p = c(\widetilde{e}_i)\nabla^{\mathbb{E}_p}_{\widetilde{e}_i}. \tag{2.4.4}$$

Thus, using elliptic estimates we know that there are $C, C', C'' > 0$ such that

$$\|s\|_{\boldsymbol{H}^2(p)} \leq C(\|D_p^2 s\|_{L^2} + p^2\|s\|^2_{L^2}). \tag{2.4.5}$$

Let Q be a differential operator of order $2m$, $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_k. Then $[D_p^2, Q]$ is a differential operator of order $m+1$. More precisely, from (2.3.44) and (2.4.4), we know that it has the following structure:

$$[D_p^2, Q] = (\text{order } 2m+1) + (\text{order } 2m)(1+p) + (\text{order } 2m-1)(p+p^2). \tag{2.4.6}$$

Thus, with (2.4.5), we have

$$\begin{aligned}\|Qs\|_{\boldsymbol{H}^2(p)} &\leq C\big(\|D_p^2 Qs\|_{L^2} + p^2\|Qs\|^2_{L^2}\big)\\ &\leq C\big(\|QD_p^2 s\|_{L^2} + p^2\|Qs\|^2_{L^2} + p^2\|s\|^2_{\boldsymbol{H}^{2m+1}(p)}\big).\end{aligned} \tag{2.4.7}$$

This entails

$$\|s\|_{\boldsymbol{H}^{m+2}(p)} \leq C\big(\|D_p^2 s\|_{\boldsymbol{H}^{2m}(p)} + p^2\|s\|^2_{\boldsymbol{H}^{2m+1}(p)}\big), \tag{2.4.8}$$

from which we get (2.4.3) by induction. □

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(t) = \begin{cases} 1 \text{ for } |t| < \varepsilon/2, \\ 0 \text{ for } |t| > \varepsilon. \end{cases} \tag{2.4.9}$$

For $u > 0$, $\varsigma \geq 1$ and $a \in \mathbb{C}$, set

$$\begin{aligned}\mathrm{F}_u(a) &= \int_{\mathbb{R}} e^{iv\sqrt{2}a} \exp(-v^2/2) f(\sqrt{u}v) \frac{dv}{\sqrt{2\pi}},\\ \mathrm{G}_u(a) &= \int_{\mathbb{R}} e^{iv\sqrt{2}a} \exp(-v^2/2)(1 - f(\sqrt{u}v)) \frac{dv}{\sqrt{2\pi}},\\ \mathrm{H}_{u,\varsigma}(a) &= \int_{\mathbb{R}} e^{iv\sqrt{2}a} \exp(-v^2/2u)(1 - f(\sqrt{\varsigma}v)) \frac{dv}{\sqrt{2\pi}}.\end{aligned} \tag{2.4.10}$$

These functions are even holomorphic functions. Moreover, the restriction of F_u and G_u to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$\mathrm{G}_{\frac{u}{p}}\left(a\sqrt{u/p}\right) = \mathrm{H}_{\frac{u}{p},1}(a) \quad \text{and} \quad \mathrm{F}_u(vD_p)+\mathrm{G}_u(vD_p) = \exp\left(-v^2D_p^2\right) \text{ for } v>0. \tag{2.4.11}$$

Let $\tilde{\mathrm{G}}_u(vL_p)(x,x')$ be the smooth kernel of $\mathrm{G}_u(vL_p)$ with respect to $dv_M(x')$.

Proposition 2.4.2. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $u > 0$ and any $p \in \mathbb{N}^*$,*

$$\left|\mathrm{G}_{\frac{u}{p}}\left(\sqrt{u/p}D_p\right)(\cdot,\cdot)\right|_{\mathscr{C}^m(M\times M)} \leq Cp^N \exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{2.4.12}$$

Here, the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E, ∇^{Bi}, h^L, h^E and g^{TM}.

Proof. As $i^m a^m e^{iva} = \frac{\partial^m}{\partial v^m} e^{iva}$, we can integrate by part the expression of $a^m \mathrm{H}_{u,\varsigma}(a)$ given in (2.4.10) to obtain that for any $m \in \mathbb{N}$ and $c > 0$, there is a $C_{m,c} > 0$ such that $u > 0$ and $\varsigma \geq 1$,

$$\sup_{a\in\mathbb{R}} |a^m \mathrm{H}_{u,\varsigma}(a)| \leq C_{m,c}\varsigma^{\frac{m}{2}} \exp\left(-\frac{\varepsilon^2}{16u\varsigma}\right). \tag{2.4.13}$$

Let Q be a differential operator of order $2m$, $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_k. Using Lemma 2.4.1 and (2.4.13), we find that for $m' \in \mathbb{N}$,

$$\begin{aligned}\left|\langle L_p^{m'}\mathrm{H}_{\frac{u}{p},1}(L_p)Qs, s'\rangle\right| &= \left|\langle s, Q^*\mathrm{H}_{\frac{u}{p},1}(L_p)L_p^{m'}s'\rangle\right| \\ &\leq C\|s\|_{L^2}\left\|\mathrm{H}_{\frac{u}{p},1}(L_p)L_p^{m'}s'\right\|_{\boldsymbol{H}^{2m}(p)} \\ &\leq C\|s\|_{L^2}p^{4m}\sum_{j=0}^m p^{-4j}\left\|L_p^j\mathrm{H}_{\frac{u}{p},1}(L_p)L_p^{m'}s'\right\|_{L^2} \\ &\leq CKp^{4m+k_m}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)\|s\|_{L^2}\|s'\|_{L^2}.\end{aligned} \tag{2.4.14}$$

Thus,

$$\left\|L_p^{m'}\mathrm{H}_{\frac{u}{p},1}(L_p^2)Qs\right\|_{L^2} \leq CKp^{4m+k_m}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)\|s\|_{L^2}. \tag{2.4.15}$$

We deduce from this estimate – and using once again Lemma 2.4.1 and (2.4.13) – that if P, Q are differential operators with scalar principal symbol of order $2m', 2m$ respectively and with compact support in U_k, U_ℓ respectively, then there is a positive constant $C_{m,m'}$ such that

$$\left\|P\mathrm{H}_{\frac{u}{p},1}(L_p^2)Qs\right\|_{L^2} \leq C_{m,m'}p^{4m+k_m}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)\|s\|_{L^2}. \tag{2.4.16}$$

By the Sobolev inequality, (2.4.11) and (2.4.16), we get Proposition 2.4.2. $\square$

Proof of Theorem 2.1.8. As 0 is a regular value of μ, there is ϵ_0 such that

$$\mu\colon M_{2\epsilon_0} := \mu^{-1}(B^{\mathfrak{g}^*}(0,2\epsilon_0)) \to B^{\mathfrak{g}^*}(0,2\epsilon_0) \tag{2.4.17}$$

is a submersion. Note that $M_{2\epsilon_0}$ is an open G-invariant subset of M.

Fix ε, ϵ_0 small enough so that $M_{2\epsilon_0} \subset U$ and $d^M(x,y) > 4\varepsilon$ if $x \in M_{\epsilon_0}$ and $y \in M \setminus U$. We set $V_{\epsilon_0} = M \setminus M_{\epsilon_0}$, which is a smooth G-manifold with boundary ∂V_{ϵ_0}. Then $M \setminus U \subset V_{\epsilon_0}$.

We denote by $D_{p,D}$ the operator D_p acting on V_{ϵ_0} with the Dirichlet boundary condition. Then $D_{p,D}$ is self-adjoint.

By [60, Sects. 2.6, 2.8] and [46, Append. D.2], we know that the wave operator $\cos(uD_{p,D})$ is well defined and its Schwartz kernel $\cos(uD_{p,D})(x,x')$ only depends on the restriction of D_p to $G\cdot B^M(x,u)\cap V_{\epsilon_0}$ and vanish if $d^M(x,x')\geq u$. Thus, by (2.4.10),

$$\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p\right)(x,x')=\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_{p,D}\right)(x,x')\qquad \text{if } x,x'\in M\setminus U. \tag{2.4.18}$$

Let $s\in\mathscr{C}^\infty(M,\mathbb{E}_p)^G$ with $\mathrm{supp}(s)\subset \mathring{V}_{\epsilon_0}$. Since D_p commutes with the G-action, we know that $D_ps\in\Omega^{0,\bullet}(M,L^p\otimes E)^G$. Moreover, from the Lichnerowicz formula (2.3.44) and the fact that $\mathrm{supp}(s)\subset\mathring{V}_{\epsilon_0}$, we get

$$\langle D_p^2s,s\rangle=\|\nabla^{\mathbb{E}_p}s\|^2_{L^2}-p\langle(\omega_d+\tau)s,s\rangle+\langle\Psi_{\mathcal{E}}s,s\rangle. \tag{2.4.19}$$

Moreover, as $s\in\Omega^{0,\bullet}(M,L^p\otimes E)^G$, (2.2.3) gives

$$\nabla^{\mathbb{E}_p}_{K^M}s=(\mathcal{L}_K+\mu^{\mathbb{E}_p}(K))s=\mu^{\mathbb{E}_p}(K)s, \tag{2.4.20}$$

and thus by (2.3.45),

$$\begin{aligned}\|\nabla^{\mathbb{E}_p}s\|^2_{L^2}\geq &C\sum_i\|\nabla^{\mathbb{E}_p}_{K_i^M}s\|^2_{L^2}=C\sum_i\|\mu^{\mathbb{E}_p}(K_i)s\|^2_{L^2}\\ &\geq Cp^2\|\,|\mu|s\|^2_{L^2}-C'\|s\|^2_{L^2}\\ &\geq C\epsilon_0^2p^2\|s\|^2_{L^2}-C'\|s\|^2_{L^2}.\end{aligned} \tag{2.4.21}$$

Thanks to (2.4.19) and (2.4.21), we have

$$\langle D_p^2s,s\rangle\geq Cp^2\|s\|^2_{L^2}. \tag{2.4.22}$$

In particular, as P_G preserve the Dirichlet boundary condition, there are $C,C'>0$ such that for $p\geq 1$,

$$\mathrm{Sp}(P_GD^2_{p,D}P_G)\subset[Cp^2,+\infty[\,. \tag{2.4.23}$$

By the elliptic estimate for the Laplacian with Dirichlet boundary condition [60, Thm. 5.1.3] and (2.4.4), we can prove that Lemma 2.4.1 still holds if we replace therein D_p by $D_{p,D}$ and take $s\in\boldsymbol{H}^{2m+2}(M,\mathbb{E}_p)\cap\boldsymbol{H}^1_0(M,\mathbb{E}_p)$. Thus, using (2.4.23) and

$$\sup_{a\geq Cp^2}|a^m\mathrm{F}_{\frac{u}{p}}(a\sqrt{u/p})|\leq C_{m,k,u}p^{-k}. \tag{2.4.24}$$

instead of (2.4.13), we find as in (2.4.16) that for any Q,Q differential operators of order $2m,2m'$ with scalar principal symbol and with support in U_i,U_j and for any $k\in\mathbb{N}$

$$\left\|Q\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_{p,D}\right)Q's\right\|_{L^2}\leq C_{m,m',u}p^{-k}\|s\|_{L^2}. \tag{2.4.25}$$

Thus, using Sobolev inequality as in the proof of Porposition 2.4.2, (2.4.11), (2.4.12) and (2.4.18), we get Theorem 2.1.8. □

2.5 Asymptotic of the heat kernel near P for a free action

We assume in this Section that G acts freely on P and $\overline{U}$.

In this section, we prove Theorem 2.1.9. In Section 2.5.1, we work near P and replace our geometric setting by a model setting, in which M is replaced by $G \times \mathbb{R}^{2n-d}$, P by $G \times \mathbb{R}^{2n-2d} \times \{0\}$ and the different bundles are trivial. We can then define a rescaled version of $\frac{1}{p}D_p^2$, and in Section 2.5.2, we prove the convergence of the heat kernel of the rescaled operator. In Section 2.5.3 we compute the limiting heat kernel to finish the proof of Theorem 2.1.9.

2.5.1 Rescaling the operator $\Phi D_p^2 \Phi^{-1}$

This section is analogous to [51, Sect. 2.6], with the necessary changes made.

We fix $x_0 \in M_G$ and $\varepsilon \in]0, \mathrm{inj}^M/4[$.

Recall that we have the following diagram:

$$\begin{tikzcd}
P = \mu^{-1}(0) \arrow[r, hook] \arrow[d, "G"] & U \arrow[d, "G"] \\
M_G \arrow[r, hook] & B
\end{tikzcd}$$

Recall that g^{T^HP} is a G-invariant and J-invariant metric on T^HP, g^{TY} is a G-invariant metric on TY and g^{JTY} is the G-invariant metric on JTY induced by J and g^{TY}. Then by (2.3.19), we can chose be a G-invariant metric g^{TM} on M such that on P:

$$g^{TM}|_P = g^{TY}|_P \oplus g^{JTY}|_P \oplus g^{T^HP}. \tag{2.5.1}$$

Let g^{T^HU} be the restriction of g^{TM} on T^HU. Let g^{TB} (resp. g^{TM_G}) be the metric on TB (resp. TM_G) induced by g^{T^HU} (resp. g^{T^HP}).

By (2.1.23) and Lemma 2.3.4, we know that

$$T^HU|_P = JTY|_P \oplus JT^HP = JTY|_P \oplus T^HP. \tag{2.5.2}$$

As a consequence, if N_G denotes the normal bundle of M_G in B, then N_G can be identified as

$$N_G \simeq (TM_G)^{\perp_{g^{TB}}} = (JTY)_B|_{M_G}, \tag{2.5.3}$$

where $(JTY)_B$ denotes the bundle over B induced by JTY.

Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}). Let P^{N_G} and P^{TM_G} be the orthogonal projections from $TB|_{M_G}$ to N_G and TM_G respectively. Set

$$\begin{aligned}
&\nabla^{N_G} = P^{N_G}(\nabla^{TB}|_{M_G})P^{N_G}, \quad && \nabla^{TM_G} = P^{TM_G}(\nabla^{TB}|_{M_G})P^{TM_G}, \\
&{}^0\nabla^{TB} = \nabla^{N_G} \oplus \nabla^{TM_G}, && A = \nabla^{TB}|_{M_G} - {}^0\nabla^{TB}.
\end{aligned} \tag{2.5.4}$$

For $W \in T_{x_0}M_G$, let $u \in \mathbb{R} \mapsto x_u = \exp^{M_G}_{x_0}(uW) \in M_G$ be the geodesic in M_G starting at x_0 with speed W. If $|W| \leq 4\varepsilon$ and $V \in N_{G,x_0}$, let $\tau_W V$ be the parallel transport of V with respect to ∇^{N_G} along to curve $u \in [0,1] \mapsto x_u = \exp^{M_G}_{x_0}(uW)$.

If $Z \in T_{x_0}B$, we decompose Z as $Z = Z^0 + Z^\perp$ with $Z^0 \in T_{x_0}M_G$ and $Z^\perp \in N_{G,x_0}$, and we identify Z with $\exp^B_{\exp^{M_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)$. This gives a diffeomorphism

$$\Psi \colon B^{T_{x_0}M_G}(0,4\varepsilon) \times B^{N_{G,x_0}}(0,4\varepsilon) \xrightarrow{\sim} \mathscr{U}(x_0) \subset B, \tag{2.5.5}$$

where $\mathscr{U}(x_0)$ is an open neighborhood of x_0 in B. Note that $\mathscr{U}(x_0)\cap M_G = B^{T_{x_0}M_G}(0,4\varepsilon)\times\{0\}$.

In the sequel, we will indifferently write $B^{T_{x_0}M_G}(0,4\varepsilon)\times B^{N_{G,x_0}}(0,4\varepsilon)$ or $\mathscr{U}(x_0)$, x_0 or 0, etc...

We identify $(L_B)_Z$, $(E_B)_Z$ and $(\mathbb{E}_p)_{B,Z}$ with $(L_B)_{x_0}$, $(E_B)_{x_0}$ and $(\mathbb{E}_p)_{B,x_0}$ by using parallel transport with respect to ∇^{L_B}, ∇^{E_B} and $\nabla^{(\mathbb{E}_p)_B}$ along the curve $u\in[0,1]\mapsto \gamma_u = uZ$.

Fix $y_0\in\pi^{-1}(x_0)$. We define $\widetilde{\gamma}\colon[0,1]\to M$ to be the curve lifting γ such that $\frac{\partial\widetilde{\gamma}_u}{\partial u}\in T^H_{\widetilde{\gamma}_u}U$. As above, on $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$, we can trivialize L, E and $\mathbb{E}_p$ using the parallel transport along $\widetilde{\gamma}$ with respect to the corresponding connections. By (2.2.6), the previous trivialization are naturally induced by this one.

This also gives a diffeomorphism

$$\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))\simeq G\times B^{T_{x_0}B}(0,4\varepsilon), \tag{2.5.6}$$

and the induced G-action on $G\times B^{T_{x_0}B}(0,\varepsilon)$ is then

$$g.(g',Z)=(gg',Z). \tag{2.5.7}$$

Let $\{e_i^0\}$ and $\{e_i^\perp\}$ be orthonormal basis of $T_{x_0}M_G$ and N_{G,x_0} respectively. Then $\{e_i\}=\{e_i^0,e_i^\perp\}$ is an orthonormal basis of $T_{x_0}B$. Let $\{e^i\}$ be its dual basis. We will also denote $\Psi_*(e_i^0)$, $\Psi_*(e_i^\perp)$ by $\{e_i^0\}$, $\{e_i^\perp\}$, so that in our coordinates,

$$\frac{\partial}{\partial Z_i^0}=e_i^0\,,\quad \frac{\partial}{\partial Z_i^\perp}=e_i^\perp. \tag{2.5.8}$$

In what follows, we will extend the geometric object from $B^{T_{x_0}B}(0,4\varepsilon)$ to $\mathbb{R}^{2n-d}\simeq T_{x_0}B$ (here the identification is similar to (2.4.1)) to get analogue geometric structures on $G\times\mathbb{R}^{2n-d}$ as on M, an thus work on

$$M_0 := G\times\mathbb{R}^{2n-d} \tag{2.5.9}$$

instead of M.

Let L_0 be the trivial bundle $L|_{G.y_0}$ lifted on M_0. We still denote by ∇^L, h^L the connection and metric on L_0 over $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$ induced by the above identification. Then h^L is identified with the constant metric $h^{L_0}=h^{L_{y_0}}$. We use similar notations for the bundle E.

Let $\varphi\colon\mathbb{R}\to[0,1]$ be a smooth even function such that

$$\varphi(v)=\begin{cases}1 \text{ for } |v|<2,\\ 0 \text{ for } |v|>4.\end{cases} \tag{2.5.10}$$

Let $\varphi_\varepsilon\colon M_0\to M_0$ defined by

$$\varphi_\varepsilon(g,Z)=(g,\varphi(|Z|/\varepsilon)Z). \tag{2.5.11}$$

Let $\nabla^{E_0}=\varphi_\varepsilon^*\nabla^E$. Then ∇^{E_0} is an extension of ∇^E outside $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$.

Let P^{TY} be the orthogonal projection from TM onto TY. For $W\in TB$, let $W^H\in T^HU$ be the horizontal lift of W. Then we can define the Hermitian connection ∇^{L_0} on $(L_0,h^{L_0})\to G\times\mathbb{R}^{2n-d}$ by

$$\nabla^{L_0}=\varphi_\varepsilon^*\nabla^L+(1-\varphi(|Z|/\varepsilon))R^L_{y_0}(Z^H,P^{TY}_{y_0}\cdot). \tag{2.5.12}$$

We can compute directly the curvature R^{L_0} of ∇^{L_0}: if we denote $(1, Z)$ just by Z, then

$$\begin{aligned} R^{L_0}_Z =& R^L_{\varphi_\varepsilon(Z)}(P^{TY}_{y_0}\cdot, P^{TY}_{y_0}\cdot) + R^L_{y_0}(P^{T^HU}_{y_0}\cdot, P^{TY}_{y_0}\cdot) \\ &+ \varphi^2(|Z|/\varepsilon) R^L_{\varphi_\varepsilon(Z)}(P^{T^HU}_{y_0}\cdot, P^{T^HU}_{y_0}\cdot) \\ &+ \varphi(|Z|/\varepsilon)\big[R^L_{\varphi_\varepsilon(Z)} - R^L_{y_0}\big](P^{T^HU}_{y_0}\cdot, P^{TY}_{y_0}\cdot) \\ &+ \varphi'(|Z|/\varepsilon)\frac{Z^*}{\varepsilon|Z|} \wedge \big[R^L_{\varphi_\varepsilon(Z)} - R^L_{y_0}\big](Z^H, P^{TY}_{y_0}\cdot) \\ &+ (\varphi\varphi')(|Z|/\varepsilon)\frac{Z^*}{\varepsilon|Z|} \wedge R^L_{\varphi_\varepsilon(Z)}(Z^H, P^{T^HU}_{y_0}\cdot), \end{aligned} \tag{2.5.13}$$

where $Z^* \in T^*_{x_0}B$ is the dual of $Z \in T_{x_0}B$ with respect to the metric $g^{TB}_{x_0}$.

The group G acts naturally on M_0 by (2.5.7) and under our identifications, the action of G on L, E on $G \times^{T_{x_0}B} (0, \varepsilon)$ is exactly the G-action on $L|_{G.y_0}$, $E|_{G.y_0}$.

We define a G-action on L_0, E_0 by the action of G on $G.y_0$. Then it extends the G-action on L, E on $G \times^{T_{x_0}B} (0, \varepsilon)$ to M_0.

By Lemma 2.3.4, we know that

$$R^L_{(1,Z^0)}(Z^H, K^M) = R^L_{(1,Z^0)}((Z^\perp)^H, K^M). \tag{2.5.14}$$

For $(1, Z) \in G \times \mathbb{R}^{2n-d}$, (2.5.7) gives $\varphi_{\varepsilon *} K^{M_0}_{(1,Z)} = K^M_{y_0}$ for $K \in \mathfrak{g}$. Thus, by (2.1.7), (2.5.12) and (2.5.14), the moment map $\mu_0 \colon M_0 \to \mathfrak{g}^*$ of the G-action on M_0 is given by

$$\mu_0(K)_{(1,Z)} = \mu(K)_{\varphi_\varepsilon(1,Z)} + \frac{1}{2i\pi}(1 - \varphi(|Z|/\varepsilon)) R^L_{y_0}((Z^\perp)^H, K^M_{y_0}). \tag{2.5.15}$$

Now, from the construction of our coordinate, we have $\mu_0 = 0$ on $G \times \mathbb{R}^{2n-2d} \times \{0\}$. Moreover,

$$\mu(K)_{\varphi_\varepsilon(1,Z)} = \frac{1}{2i\pi} R^L_{(1,Z)}(\varphi(|Z|/\varepsilon)(Z^\perp)^H, K^M) + O(\varphi(|Z|/\varepsilon)|Z||Z^\perp|). \tag{2.5.16}$$

Thus, from our construction, Lemma 2.3.3 and (2.5.3), (2.5.15) and (2.5.16), we know that

$$\mu_0^{-1}(0) = G \times \mathbb{R}^{2n-2d} \times \{0\}. \tag{2.5.17}$$

Let

$$g^{TM_0}(g, Z) = g^{TM}(\varphi_\varepsilon(g, Z)) \quad \text{and} \quad J_0(g, Z) = J(\varphi_\varepsilon(g, Z)) \tag{2.5.18}$$

be the metric and almost-complex structure on M_0. Let $T^{*(0,1)}M_0$ be the anti-holomorphic cotangent bundle of (M_0, J_0). Since $J_0(g, Z) = J(\varphi_\varepsilon(g, Z))$, $T^{*(0,1)}_{(g,Z),J_0}M_0$ is naturally identified with $T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}M_0$.

We can now construct all the objects corresponding to those of Section 2.3.2 in this new setting and denotes them by adding subscripts 0, e.g. $\mathbb{E}_{0,p}$, $\nabla^{\det_0}$, ∇^{Cl_0}, ∇^{Bi_0}, $\nabla^{\mathbb{E}_{0,p}}$, ... Then, we can define the Dirac operator $D^{M_0}_p$ on M_0, which satisfies

$$D^{M_0,2}_p = \Delta^{\mathbb{E}_{0,p}} - p(2\omega_{d,0} + \tau_0) + \Psi_{\mathcal{E}_0}. \tag{2.5.19}$$

By (2.3.44) and the above constructions, we know that D^2_p and $D^{M_0,2}_p$ coincide on $\pi^{-1}(B^{T_{x_0}B}(0, 2\varepsilon))$.

We can identify $\Lambda^{0,\bullet}(T^*_{(g,Z)}M_0)$ with $\Lambda^{0,\bullet}(T^*_{gy_0}M)$ by identifying first $\Lambda^{0,\bullet}(T^*_{(g,Z)}M_0)$ with $\Lambda^{0,\bullet}(T^*_{\varphi_\varepsilon(g,Z),J}M)$ and then identifying $\Lambda^{0,\bullet}(T^*_{\varphi_\varepsilon(g,Z),J}M)$ with $\Lambda^{0,\bullet}(T^*_{gy_0}M)$ by parallel transport with respect to ∇^{Bi_0} (see (2.3.39)) along $u \in [0,1] \mapsto (g, u\varphi(|Z|/\varepsilon)Z)$. We also trivialize $\det(T^{(1,0)}M_0)$ in this way using $\nabla^{\det_0}$.

Let g^{TB_0} be the metric on $B_0 = \mathbb{R}^{2n-d}$ induced by g^{TM_0}, and let dv_{B_0} by the corresponding Riemannian volume. We denote by TY_0 the relative tangent bundle of the fibration $M_0 \to B_0$, and by g^{TY_0} the metric on TY_0 induced by g^{TM_0}.

The operator $\Phi D_p^{M_0,2}\Phi^{-1}$ is also well-defined on $T_{x_0}B \simeq \mathbb{R}^{2n-d}$. More precisely, it is an operator on the bundle $(\mathbb{E}_{0,p})_{B_0}$ over B_0 induced by $\mathbb{E}_{0,p}$, and by (2.3.46), it is given by

$$\Phi D_p^{M_0,2}\Phi^{-1} = \Delta^{(\mathbb{E}_{0,p})_{B_0}} - p(2\omega_{0,d} + \tau_0) + \Psi_{\mathcal{E}_0} - \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}} \rangle_{g^{TY_0}} - \frac{1}{h_0}\Delta_{B_0} h_0. \quad (2.5.20)$$

Let $\exp(-uD_p^{M_0,2})(Z,Z')$ be the smooth heat kernel of $D_p^{M_0,2}$ with respect to $dv_{M_0}(Z')$.

Lemma 2.5.1. *Under notation of Proposition 2.4.2 and the above trivializations, the following estimate holds uniformly on $v = (g,Z), v' = (g',Z') \in G \times B^{T_{x_0}B}(0,\varepsilon)$:*

$$\left| e^{-\frac{u}{p}D_p^2}(v,v') - e^{-\frac{u}{p}D_p^{M_0,2}}((g,Z),(g',Z')) \right| \leq Cp^N \exp\big(-\frac{\varepsilon^2 p}{16u}\big), \quad (2.5.21)$$

Proof. By (2.5.19), $D_p^{M_0,2}$ has the same structure as D_p^2. Thus Lemma 2.4.1 and Proposition 2.4.2 are still true if we replace D_p^2 therein by $D_p^{M_0,2}$. Moreover, as $D_p^{M_0,2}$ and D_p^2 coincide for $|Z|$ small and the finite propagation speed of the wave equation (see e.g. [46, Thm. D.2.1]), we know that

$$\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p\right)(v,\cdot) = \mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p^{M_0}\right)((g,Z),\cdot) \quad (2.5.22)$$

if $v = (g,Z)$ under the above trivializations. Thus, we get our Lemma by (2.4.11). □

We still denote P_G the orthogonal projection from $\Omega^{0,\bullet}(U, L^p \otimes E)$ onto $\Omega^{0,\bullet}(U, L^p \otimes E)^G$. Let dg be the Haar measure on G. Then we have

$$(P_G e^{-\frac{u}{p}D_p^2} P_G)(v,v') = \int_{G\times G} (g,g') \cdot e^{-\frac{u}{p}L_p}(g^{-1}v, g'v')dg dg'. \quad (2.5.23)$$

If we again denote by P_G the orthogonal projection from $\Omega^{0,\bullet}(M_0, L_0^p \otimes E_0)$ onto $\Omega^{0,\bullet}(M_0, L_0^p \otimes E_0)^G$, then we have a similar formula as (2.5.23) for $(P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G)$. Thus, as G preserve every metrics and connections, Lemma 2.5.1 implies

Corollary 2.5.2. *Under notation of Proposition 2.4.2, the following estimate holds uniformly on $v = (g,Z), v' = (g',Z') \in G \times B^{T_{x_0}B}(0,\varepsilon)$:*

$$\left| (P_G e^{-\frac{u}{p}D_p^2} P_G)(v,v) - (P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G)((g,Z),(g',Z')) \right| \leq Cp^N \exp\big(-\frac{\varepsilon^2 p}{16u}\big), \quad (2.5.24)$$

Let S_L be a G-invariant unit section of $L|_{Gy_0}$. Let pr be the projection $G \times \mathbb{R}^{2n-d} \to G$. Using S_L and the above discussion, we get two isometries

$$\mathbb{E}_{0,p} = \Lambda^{0,\bullet}(T^*M_0) \otimes E_0 \otimes L_0^p \simeq \mathrm{pr}^*(\mathcal{E}|_{Gy_0}) \quad \text{and} \quad (\mathbb{E}_{0,p})_{B_0} \simeq \mathcal{E}_{B,x_0}. \quad (2.5.25)$$

Thus, $\Phi D_p^{M_0,2}\Phi^{-1}$ can be seen as an operator on $\mathcal{E}_{B,x_0}$. Note that our formulas will not depend on the choice of S_L as the isomorphism $\mathrm{End}((\mathbb{E}_{0,p})_{B_0}) \simeq \mathrm{End}(\mathcal{E}_{B,x_0})$ is canonical.

Let dv_{TB} be the Riemannian volume of $(T_{x_0}B, g^{TB})$. Recall that κ is the smooth positive function defined by

$$dv_{B_0}(Z) = \kappa(Z) dv_{TB}(Z) = \kappa(Z) dv_{M_G}(x_0) dv_{N_{G,x_0}}, \tag{2.5.26}$$

with $\kappa(0) = 1$.

As in (2.2.7), we denote by R^{L_B}, R^{E_B} and R^{Bi_B} the curvature on L_B, E_B and $\left(\Lambda^{0,\bullet}(T^*M)\right)_B$ induced by ∇^L, ∇^E and ∇^{Bi} on M.

As in (2.2.4), $\widetilde{\mu} \in TY$, $\widetilde{\mu}^E \in TY \otimes \mathrm{End}(E)$ and $\widetilde{\mu}^{\mathrm{Bi}} \in TY \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*M))$ are the sections induced by μ, μ^E and μ^{Bi} in (2.2.3) and (2.3.45).

We denote by ∇_V the ordinary differentiation operator on $T_{x_0}B = B_0$ in the direction V.

We will now make the change of parameter $t = \frac{1}{\sqrt{p}} \in]0,1]$.

Definition 2.5.3. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n-d}, \mathcal{E}_{y_0})$ and $Z \in \mathbb{R}^{2n-d}$ set

$$\begin{aligned}
&(S_t s)(Z) = s(Z/t),\\
&\nabla_t = t S_t^{-1} \kappa^{1/2} \nabla^{(\mathbb{E}_{0,p})_{B_0}} \kappa^{-1/2} S_t,\\
&\nabla_0 = \nabla + \frac{1}{2} R^{L_B}_{x_0}(Z, \cdot),\\
&\mathscr{L}_t = t^2 S_t^{-1} \kappa^{1/2} \Phi D_p^{M_0,2} \Phi^{-1} \kappa^{-1/2} S_t,\\
&\mathscr{L}_0 = -\frac{1}{2} \sum_i (\nabla_{0,e_i})^2 - 2\omega_{d,x_0} - \tau_{x_0} + 4\pi^2 |P^{TY} \mathbf{J}_{x_0} Z|^2.
\end{aligned} \tag{2.5.27}$$

Proposition 2.5.4. *When $t \to 0$, we have*

$$\nabla_{t,e_i} = \nabla_{0,e_i} + O(t) \ \textit{and} \quad \mathscr{L}_t = \mathscr{L}_0 + O(t). \tag{2.5.28}$$

Proof. Let Γ^{L_B}, Γ^{E_B} and Γ^{Bi_B} be the connection form of ∇^{L_B}, ∇^{E_B} and ∇^{Bi_B} with respect to fixed frame of L_B, E_B and $\left(\Lambda^{0,\bullet}(T^*M)\right)_B$ which are parallel along the curve $u \in [0,1] \mapsto uZ$ under our trivialization on $B^{T_{x_0}B}(0, 4\varepsilon)$.

By (2.5.27), we have for $|Z| \leq \varepsilon/t$

$$\nabla_{t,e_i}(Z) = \kappa^{1/2}(tZ) \left\{ \nabla_{e_i} + \left(t^{-1} \Gamma^{L_B}_{tZ}(e_i) + t \Gamma^{E_B}_{tZ}(e_i) + t \Gamma^{\mathrm{Bi}_B}_{tZ}(e_i) \right) \right\} \kappa^{-1/2}(tZ). \tag{2.5.29}$$

It is a well known fact (see for instance [46, Lemma 1.2.4]) that for if $\Gamma = \Gamma^{L_B}$ (resp. Γ^{E_B}, Γ^{Bi_B}) and $R = R^L$ (resp. R^{E_B}, R^{Bi_B}), then

$$\Gamma_Z(e_i) = \frac{1}{2} R_{x_0}(Z, e_i) + O(|Z|^2). \tag{2.5.30}$$

Thus,

$$\begin{aligned}
&t \Gamma^{E_B}_{tZ}(e_i) + t \Gamma^{\mathrm{Bi}_B}_{tZ}(e_i) = O\left(t^2\right),\\
&t^{-1} \Gamma^L_{tZ}(e_i) = \frac{1}{2} R^L_{x_0}(Z, e_i) + O(t).
\end{aligned} \tag{2.5.31}$$

The first asymptotic development in Proposition 2.5.4 follows from $\varphi(0) = \kappa(0) = 1$, (2.5.29), (2.5.30) and (2.5.31).

Let $(g^{ij}(Z))$ is the inverse of the matrix $(g_{ij}(Z)) := (g^{T_{x_0}B}_Z(e_i, e_j))$. By (2.3.41), (2.5.20) and (2.5.27) we have

$$\begin{aligned}
\mathscr{L}_t(Z) = -g^{ij}(tZ) \left(\nabla_{t,e_i} \nabla_{t,e_j} - t \nabla_{t, \nabla^{TB_0}_{e_i} e_j} \right) - \langle t \widetilde{\mu}^{\mathbb{E}_{0,p}}, t \widetilde{\mu}^{\mathbb{E}_{0,p}} \rangle_{g^{TY}}(tZ)\\
- (2\omega_{0,d} + \tau_0)(tZ) + t^2 \Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0} \Delta_{B_0} h_0 \Big)(tZ).
\end{aligned} \tag{2.5.32}$$

With the asymptotic of ∇_t above, (2.3.41) and the fact that $g^{ij}(0) = \delta_{ij}$ we find

$$-g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TB_0}_{e_i}e_j}\right) = \sum_i (\nabla_{0,e_i})^2 + O(t). \tag{2.5.33}$$

Moreover,

$$-(2\omega_{0,d} + \tau_0)(tZ) + t^2\Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0}\Delta_{B_0}h_0\Big)(tZ) = -2\omega_{d,x_0} - \tau_{x_0} + O(t). \tag{2.5.34}$$

Now, by (2.3.1), (2.1.25) and the fact that $\widetilde{\mu}_{y_0} = 0$ for $y_0 \in P$, $\pi(y_0) = x_0$, we get for $K \in \mathfrak{g}$:

$$-\langle \mathbf{J}e_i^H, K^M\rangle_{y_0} = \omega(K^M, e_i^H) = \nabla_{e_i^H}(\mu(K)) = \langle \nabla^{TY}_{e_i^H}\widetilde{\mu}, K^M\rangle_{y_0}. \tag{2.5.35}$$

Thus,

$$|\widetilde{\mu}|^2_{g^{TY}}(Z) = |\nabla^{TY}_Z\widetilde{\mu}|^2_{g^{TY}} + O(|Z|^3) = |P^{TY}\mathbf{J}_{x_0}Z|^2_{g^{TY}} + O(|Z|^3). \tag{2.5.36}$$

Note that

$$\langle t\widetilde{\mu}^{\mathbb{E}_p}, t\widetilde{\mu}^{\mathbb{E}_p}\rangle_{g^{TY}} = -4\pi^2\frac{1}{t^2}|\widetilde{\mu}|^2_{g^{TY}} + \langle 4i\pi\widetilde{\mu} + t^2(\widetilde{\mu}^E + \widetilde{\mu}^{\mathrm{Bi}}), \widetilde{\mu}^E + \widetilde{\mu}^{\mathrm{Bi}}\rangle_{g^{TY}}. \tag{2.5.37}$$

Thus, we get the second asymptotic development in Proposition 2.5.4 by using (2.5.32), (2.5.33), (2.5.34), (2.5.36) and (2.5.37). □

2.5.2 Convergence of the heat kernel

In this section, we prove the convergence of the heat kernel of the rescaled operator. Note that here we must have a more precise result than in [46, Sect. 1.6] because in the proof of Theorem 2.1.3 (see Section 2.6.1) we will have to integrate along the normal directions, and thus we need a result of decay in these directions. To obtain it, we draw our inspiration form [51].

Recall that $\mathcal{E}_0 = \Lambda^{0,\bullet}(T^*M_0) \otimes E_0$ and that we have trivialized the Hermitian bundle $(\mathcal{E}_{0,B_0}, h^{\mathcal{E}_{0,B_0}})$ on $B_0 = T_{x_0}B$ by identifying it to $(\mathcal{E}_{B,x_0}, h^{\mathcal{E}_{B,x_0}})$. Recall also that $\mu_0 \colon M_0 \to \mathfrak{g}^*$ is the moment map of the G-action on M_0.

Let $\|\cdot\|_{L^2}$ be the L^2-norm on $\mathscr{C}^\infty(B_0, \mathcal{E}_{B,x_0})$ induced by $g^{T_{x_0}B}$ and $h^{\mathcal{E}_{B,x_0}}$ as in (2.2.8).

Let $\{f_l\}$ be a G-invariant orthonormal frame of TY on $\pi^{-1}(B^B(x_0, 4\varepsilon))$, then $\{f_{0,l}(Z) = f_l(\varphi_\varepsilon(g,Z))\}$ is a G-invariant orthonormal frame of TY_0 on M_0.

Definition 2.5.5. Set

$$\mathcal{D}_t = \Big\{\nabla_{t,e_i}\,,\ 1 \le i \le 2n-d\,;\ \frac{1}{t}\langle\widetilde{\mu}_0, f_{0,l}\rangle(tZ)\,,\ 1 \le l \le d\Big\}, \tag{2.5.38}$$

and for $k \in \mathbb{N}^*$, let $\mathcal{D}_t^m$ be the family of operators Q acting on $\mathscr{C}^\infty(T_{x_0}B, \mathcal{E}_{B,x_0})$ which can be written in the form $Q = Q_1 \dots Q_m$ with $Q_i \in \mathcal{D}_t$.

For $s \in \mathscr{C}^\infty(B_0, \mathcal{E}_{B,x_0})$ and $k \in \mathbb{N}^*$, set

$$\begin{aligned} ||s||^2_{t,0} &= ||s||^2_{L^2}, \\ ||s||^2_{t,m} &= ||s||^2_{t,0} + \sum_{\ell=1}^m \sum_{Q \in \mathcal{D}_t^\ell} ||Qs||^2_{t,0}. \end{aligned} \tag{2.5.39}$$

We denote by $\boldsymbol{H}_t^m$ the Sobolev space $\boldsymbol{H}^m(B_0, \mathcal{E}_{B,x_0})$ endowed with the norm $||\cdot||_{t,m}$, and by $\boldsymbol{H}_t^{-1}$ the Sobolev space of order -1 endowed with the norm

$$||s||_{t,-1} = \sup_{s' \in \boldsymbol{H}_p^1 \setminus \{0\}} \frac{\langle s, s' \rangle_{t,0}}{||s'||_{t,0}} . \tag{2.5.40}$$

Finally, if $A \in \mathscr{L}(\boldsymbol{H}_t^k, \boldsymbol{H}_t^m)$, we denote by $||A||_t^{k,m}$ the operator norm of A associated with $||\cdot||_{t,k}$ and $||\cdot||_{t,m}$.

Then $\mathscr{L}_t$ is a formally self-adjoint elliptic operator with respect to $\|\cdot\|_{t,0}$ and is a smooth family of operators with respect to the parameter $x_0 \in M_G$.

We denote by $\mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$ the set of smooth section of $\mathcal{E}_{B,x_0}$ over B_0 with compact support.

Proposition 2.5.6. *There exist constants $C_1, C_2, C_3 > 0$ such that for any $t \in]0,1]$ and any $s, s' \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,*

$$\begin{aligned} \langle \mathscr{L}_t s, s \ \rangle_{t,0} &\geq C_1 ||s||_{t,1}^2 - C_2 ||s||_{t,0}^2, \\ \big|\langle \mathscr{L}_t s, s' \ \rangle_{t,0}\big| &\leq C_3 ||s||_{t,1} ||s'||_{t,1}. \end{aligned} \tag{2.5.41}$$

Proof. From (2.5.32) and (2.5.39), we have

$$\begin{aligned} \langle \mathscr{L}_t s, s\rangle_{t,0} = \|\nabla_t s\|_{t,0}^2 - t^2 \big\langle \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}} \rangle_{g^{TY}}(tZ) s, s \big\rangle_{t,0} \\ + \left\langle S_t^{-1}\Big(-(2\omega_{0,d} + \tau_0) + t^2\Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0}\Delta_{B_0} h_0\Big)\Big) s, s \right\rangle_{t,0} . \end{aligned} \tag{2.5.42}$$

By (2.5.15) and our constructions, we know that for $Z \in T_{\mathbb{R},x_0}B$ with $|Z| > 4\varepsilon$,

$$\mu^{\mathbb{E}_{0,p}}(K)_{(1,Z)} = 2i\pi p\, \mu_0(K)_{(1,Z)} = p R_{y_0}^L\big((Z^\perp)^H, K_{y_0}^X\big). \tag{2.5.43}$$

Thus, from (2.2.12), (2.5.15), (2.5.37) and (2.5.43), we get

$$- t^2 \big\langle \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}} \rangle_{g^{TY}}(tZ) s, s \big\rangle_{t,0} \geq 2\pi^2 \sum_{l=1}^d \Big\| \frac{1}{t} \langle \widetilde{\mu}_0, f_{0,l} \rangle (tZ) s \Big\|_{t,0}^2 - Ct\|s\|_{t,0}^2. \tag{2.5.44}$$

Now, (2.5.41) follows from (2.5.42) and (2.5.44). □

Let Γ be the contour in $\mathbb{C}$ defined in Figure 2.1.

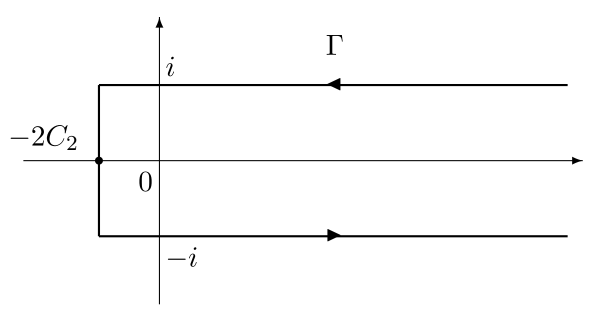

Figure 2.1: The contour Γ

Proposition 2.5.7. *There exist $t_0 > 0$ and $C > 0$, $a, b \in \mathbb{N}$ such that for any $t \in]0, t_0]$ and any $\lambda \in \Gamma$, the resolvant $(\lambda - \mathscr{L}_t)^{-1}$ exists and*

$$\begin{aligned} \left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{0,0} &\le C, \\ \left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{-1,1} &\le C(1 + |\lambda|^2). \end{aligned} \tag{2.5.45}$$

Proof. Note that $\mathscr{L}_t$ is self-adjoint operator, thus (2.5.41) implies that $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \Gamma$ and there is a constant $C > 0$ (independent on λ) such that

$$\left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{0,0} \le C. \tag{2.5.46}$$

On the other hand, if $\lambda_0 \in] - \infty, -2C_2]$, then (2.5.41) also implies that

$$\left\|(\lambda_0 - \mathscr{L}_t)^{-1}\right\|_t^{-1,1} \le \frac{1}{C_1}. \tag{2.5.47}$$

Then, using the fact that

$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda_0 - \mathscr{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t)^{-1}(\lambda_0 - \mathscr{L}_t)^{-1}, \tag{2.5.48}$$

we find that

$$\left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{-1,0} \le \frac{1}{C_1}\left(1 + C|\lambda - \lambda_0|\right). \tag{2.5.49}$$

Finally, exchanging the last two factors in (2.5.48) and applying (2.5.49), we get

$$\begin{aligned} \left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{-1,1} &\le \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{C_1^2}\left(1 + C|\lambda - \lambda_0|\right) \\ &\le C(1 + |\lambda|^2). \end{aligned} \tag{2.5.50}$$

The proof of our Proposition is complete. □

Proposition 2.5.8. *Take $m \in \mathbb{N}^*$. Then there exists a contant $C_m > 0$ such that for any $t \in]0, 1]$, $Q_1, \ldots, Q_m \in \mathcal{D}_t \cup \{Z_i\}_{i=1}^{2n-d}$ and $s, s' \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,*

$$\left|\left\langle [Q_1, [Q_2, \ldots [Q_m, \mathscr{L}_t] \ldots]]s, s'\right\rangle_{t,0}\right| \le C_m ||s||_{t,1}||s'||_{t,1}. \tag{2.5.51}$$

Proof. First, note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$. Thus by (2.5.32), we know that $[Z_j, \mathscr{L}_t]$ satisfies (2.5.51).

Using (2.5.15) and (2.5.43), we see that $\left(\nabla_{e_i}\langle\widetilde{\mu}_0, f_{0,l}\rangle\right)(tZ)$ is uniformly bounded with its derivatives for $t \in [0, 1]$, and for $|Z| \ge 4\varepsilon$,

$$\left(\nabla_{e_i}\langle\widetilde{\mu}_0, f_{0,l}\rangle\right)(Z) = \left(e_i\langle\widetilde{\mu}_0, f_{0,l}\rangle\right)_{x_0} = \omega_{x_0}(f_{0,l}, e_i). \tag{2.5.52}$$

Thus, $[\frac{1}{t}\langle\widetilde{\mu}_0, f_{0,l}\rangle(tZ), \mathscr{L}_t]$ also satisfies (2.5.51).

Let $R^{(L_0)_{B_0}}$ and $R^{(\mathcal{E}_0)_{B_0}}$ be the curvatures of the connections on $(L_0)_{B_0}$ and $(\mathcal{E}_0)_{B_0}$ induced by ∇^{L_0}, ∇^{E_0} and ∇^{Bi_0}. Then by (2.5.27), we have

$$[\nabla_{t,e_i}, \nabla_{t,e_j}] = \left(R^{(L_0)_{B_0}} + t^2 R^{(\mathcal{E}_0)_{B_0}}\right)_{tZ}(e_i, e_j). \tag{2.5.53}$$

By (2.5.32), (2.5.52) and (2.5.53), we find that $[\nabla_{t,e_i}, \mathscr{L}_t]$ has the same structure as $\mathscr{L}_t$ for $t \in]0,1]$, by which we mean that it is of the form

$$\sum_{i,j} a_{ij}(t,tZ)\nabla_{t,e_i}\nabla_{t,e_i} + \sum_i b_i(t,tZ)\nabla_{t,e_i} + c(t,tZ) \\ + \sum_l \left[d_l(t,tZ)\frac{1}{t}\langle \widetilde{\mu}_0, f_{0,l}\rangle(tZ) + d'\left|\frac{1}{t}\widetilde{\mu}_0\right|^2_{g^{TY}} \right], \quad (2.5.54)$$

where $d' \in \mathbb{C}$, and a_{ij}, b_i, c and d_l are polynomials in the first variable, and have all their derivatives in the second variable uniformly bounded for $Z \in \mathbb{R}^{2n-d}$ and $t \in [0,1]$. Note that in fact, for $[\nabla_{t,e_i}, \mathscr{L}_t]$, $d' = 0$ in (2.5.54).

The adjoint connection $(\nabla_t)^*$ of ∇_t with respect to $\langle \cdot , \cdot \rangle_{t,0}$ is given by

$$(\nabla_t)^* = -\nabla_t - t\big(\kappa^{-1}\nabla\kappa\big)(tZ). \quad (2.5.55)$$

Note that the last term of (2.5.55) and all its derivative in Z are uniformly bounded for $Z \in \mathbb{R}^{2n-d}$ and $t \in [0,1]$. Thus, by (2.5.54) and (2.5.55), we find that (2.5.51) holds when $m = 1$.

Finally, we can prove by induction that $[Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]$ has also the same structure as in (2.5.54), and thus satisfies (2.5.51) thanks to (2.5.55). □

Proposition 2.5.9. *For any $t \in]0, t_0]$, $\lambda \in \Gamma$ and $m \in \mathbb{N}$,*

$$(\lambda - \mathscr{L}_t)^{-1}(\boldsymbol{H}_t^m) \subset \boldsymbol{H}_t^{m+1}. \quad (2.5.56)$$

Moreover, for any $\alpha \in \mathbb{N}^{2n-d}$, there exist $K \in \mathbb{N}$ and $C_{\alpha,m} > 0$ such that for any $t \in]0,1]$, $\lambda \in \Gamma$ and $s \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,

$$\left\| Z^\alpha(\lambda - \mathscr{L}_t)^{-1}s \right\|_{t,m+1} \leq C_{\alpha,m}(1+|\lambda|^2)^K \sum_{\alpha' \leq \alpha} ||Z^{\alpha'}s||_{t,m}. \quad (2.5.57)$$

Proof. Let $Q_1, \dots, Q_m \in \mathcal{D}_t$ and $Q_{m+1}, \dots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n}$. Then we can express the operator $Q_1 \dots Q_{m+|\alpha|}(\lambda - \mathscr{L}_t)^{-1}$ as a linear combination of operators of the type

$$[Q_1, [Q_2, \dots [Q_\ell, (\lambda - \mathscr{L}_t)^{-1}] \dots]]Q_{\ell+1} \dots Q_{m+|\alpha|} \quad \text{with} \quad \ell \leq m + |\alpha|. \quad (2.5.58)$$

We denote by $\mathscr{F}_t$ the family of operator $\mathscr{F}_t = \{[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_k}, \mathscr{L}_t] \dots]]\}$. Then any commutator $[Q_1, [Q_2, \dots [Q_\ell, (\lambda - \mathscr{L}_t)^{-1}] \dots]]$ can be expressed as a linear combination operators of the form

$$(\lambda - \mathscr{L}_t)^{-1}F_1(\lambda - \mathscr{L}_t)^{-1}F_2 \dots F_\ell(\lambda - \mathscr{L}_t)^{-1} \quad \text{with} \quad F_j \in \mathscr{F}_t. \quad (2.5.59)$$

Moreover, by Proposition 2.5.8, the norm $\| \cdot \|_t^{1,-1}$ of any element of $\mathscr{F}_t$ is uniformly bounded by C. As a consequence, using Proposition 2.5.7 we see that there is $C > 0$ and $N \in \mathbb{N}$ such that the $\| \cdot \|_t^{0,1}$-norm of operators in (2.5.59) is bounded by $C(1+|\lambda|^2)^N$. Thus, Proposition 2.5.9 holds. □

Let $e^{-\mathscr{L}_t}(Z, Z')$ be the smooth kernel of the operator $e^{-\mathscr{L}_t}$ with respect to $dv_{TB}(Z')$. Let $\pi_{M_G} \colon TB \times_{M_G} TB \to M_G$ be the projection from the fiberwise product $TB \times_{M_G} TB$ onto M_G. As $\mathscr{L}_t$ depends on the parameter $x_0 \in M_G$, then $e^{-\mathscr{L}_t}(\cdot, \cdot)$ can be viewed as a section of $\pi_{M_G}^*(\mathrm{End}(\mathcal{E}_B))$ over $TB \times_{M_G} TB$.

Let $\nabla^{\pi_{M_G}^* \mathrm{End}(\mathcal{E}_B)}$ be the connection on $\pi_{M_G}^* \mathrm{End}(\mathcal{E}_B)$ induced by $\nabla^{\mathcal{E}_B}$. Then $\nabla^{\pi_{M_G}^* \mathrm{End}(\mathcal{E}_B)}$, h^E and g^{TM} induce naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in M_G$ on sections of $\pi_{M_G}^* (\mathrm{End}(\mathcal{E}_B))$.

As above, we will decompose any $Z \in T_{x_0}B$ as $Z = Z^0 + Z^{\perp}$, with $Z_0 \in T_{x_0}M_G$ and $Z^{\perp} \in N_{G,x_0}$.

Theorem 2.5.10. *There exists $C' > 0$ such that for any $m, m', m'', r \in \mathbb{N}$ and $u_0 > 0$, there is $C > 0$ such that for any $t \in]0, t_0]$, $u \geq u_0$ and $Z, Z' \in T_{x_0}B = B_0$*

$$\sup_{|\alpha|,|\alpha'|\leq m} (1 + |Z^{\perp}| + |Z'^{\perp}|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} e^{-u\mathscr{L}_t}(Z, Z') \right|_{\mathscr{C}^{m'}(M_G)}$$
$$\leq C\big(1 + |Z^0| + |Z'^0|\big)^{2(n+r+m'+1)+m} \exp\Big(4C_2 u - \frac{C'}{u}|Z - Z'|^2\Big), \quad (2.5.60)$$

where $|\cdot|_{\mathscr{C}^{m'}(M)}$ denotes the $\mathscr{C}^m$-norm for the parameter $x_0 \in M_G$.

Proof. By (2.5.45), we know that for $k \in \mathbb{N}^*$,

$$e^{-u\mathscr{L}_t} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_{\Gamma} e^{-u\lambda}(\lambda - \mathscr{L}_t)^{-k} d\lambda. \quad (2.5.61)$$

Then for $m \in \mathbb{N}$, we know from Proposition 2.5.9 that for $Q \in \cup_{\ell=1}^m \mathcal{D}_t^{\ell}$, there are $C_m > 0$ and $M \in \mathbb{N}$ such that for $\lambda \in \Gamma$,

$$\|Q(\lambda - \mathscr{L}_t)^{-m}\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^M. \quad (2.5.62)$$

Moreover, taking the adjoint of (2.5.62), we deduce

$$\|(\lambda - \mathscr{L}_t)^{-m} Q\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^M. \quad (2.5.63)$$

From (2.5.61), (2.5.62) and (2.5.63), we have for $Q, Q' \in \cup_{\ell=1}^m \mathcal{D}_t^{\ell}$:

$$\left\| Q e^{-u\mathscr{L}_t} Q' \right\|_t^{0,0} \leq C_m e^{2C_2 u}. \quad (2.5.64)$$

Let $\|\cdot\|_m$ be the usual Sobolev norm on $\mathscr{C}^{\infty}(T_{x_0}B, \mathcal{E}_{x_0})$ induced by $h^{\mathcal{E}_{x_0}}$ and the volume form $dv_{TX}(Z)$:

$$||s||_m^2 = \sum_{\ell\leq m} \sum_{i_1,\ldots,i_\ell} ||\nabla_{e_{i_1}} \cdots \nabla_{e_{i_\ell}} s||_0^2. \quad (2.5.65)$$

Then by (2.5.29) and (2.5.39), for any $m \in \mathbb{N}$ there exists $C'_m > 0$ such that for $s \in \mathscr{C}^{\infty}(T_{x_0}B, \mathcal{E}_{x_0})$ with support in $B^{T_{x_0}B}(0, q)$ and $t \in [0, 1]$,

$$\frac{1}{C'_m(1+q)^m} \|s\|_{t,m} \leq \|s\|_m \leq C'_m(1+q)^m \|s\|_{t,m}. \quad (2.5.66)$$

From (2.5.64), (2.5.66) and Sobolev inequalities (for $\|\cdot\|_m$) we find that if $Q, Q' \in \cup_{\ell=1}^m \mathcal{D}_t^{\ell}$, then

$$\sup_{|Z|,|Z'|\leq q} \left| Q_Z Q'_{Z'} e^{-u\mathscr{L}_t}(Z, Z') \right| \leq C(1+q)^{2n+2} e^{2C_2 u}. \quad (2.5.67)$$

Moreover, by Lemma 2.3.3 and (2.5.15), (2.5.16) and (2.5.43), we have

$$\sum_{l=1}^{d} \left| \frac{1}{t} \langle \widetilde{\mu}_0, f_{0,l} \rangle (tZ) \right|^2 \geq C|Z^{\perp}|. \quad (2.5.68)$$

Thus, (2.5.29), (2.5.67) and (2.5.68) imply (2.5.60) with the exponential e^{2C_2u} for the case where $r = m' = 0$ and $C' = 0$, i.e., for any $m, m'' \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, $Z, Z' \in T_{x_0}B = B_0$

$$\sup_{|\alpha|,|\alpha'|\leq m} \left(1+|Z^\perp|+|Z'^\perp|\right)^{m''} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} e^{-u\mathscr{L}_t}(Z,Z')\right| \\ \leq C\left(1+|Z^0|+|Z'^0|\right)^{2n+2+m} \exp(2C_2u). \quad (2.5.69)$$

To obtain the right exponential factor in the right hand side of (2.5.60), we proceed as in the proof of [7, Thm. 11.14] (see also [46, Thm. 4.2.5]).

Recall that the function f is defined in (2.4.9). For $\varsigma > 1$ and $a \in \mathbb{C}$, set

$$K_{u,\varsigma}(a) = \int_{\mathbb{R}} e^{iv\sqrt{2u}a} \exp(-v^2/2)\left(1 - f(\sqrt{2u}v/\varsigma)\right)\frac{dv}{\sqrt{2\pi}}. \quad (2.5.70)$$

Then there are $C'', C_1 > 0$ such that for any $c > 0$ and $m, m' \in \mathbb{N}$, there is $C > 0$ such that for $u \geq u_0$, $\varsigma > 1$ and $a \in \mathbb{C}$ with $|\mathrm{Im}(a)| \leq c$, we have

$$|a|^m |K_{u,\varsigma}^{(m')}(a)| \leq C \exp\left(C''c^2u - \frac{C_1}{u}\varsigma^2\right). \quad (2.5.71)$$

For $c > 0$, let V_c be the image of $\{a \in \mathbb{C} \,:\, |\mathrm{Im}(a)| \leq c\}$ by the map $a \mapsto a^2$, that is

$$V_c = \left\{\lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda) \geq \frac{1}{4c^2}\mathrm{Im}(\lambda) - c^2\right\}. \quad (2.5.72)$$

Then the contour Γ of Figure 2.1 satisfies $\Gamma \subset V_c$ for c large enough.

As $K_{u,\varsigma}$ is even, there exist a unique holomorphic function $\widetilde{K}_{u,\varsigma}$ such that $\widetilde{K}_{u,\varsigma}(a^2) = K_{u,\varsigma}(a)$. By (2.5.71), we have for $\lambda \in V_c$

$$|\lambda|^m |\widetilde{K}_{u,\varsigma}^{(m')}(\lambda)| \leq C \exp\left(C''c^2u - \frac{C_1}{u}\varsigma^2\right). \quad (2.5.73)$$

Using the finite propagation speed of the wave equation and (2.5.70), we know that there exists $c' > 0$ such that for any $\varsigma > 1$

$$\widetilde{K}_{u,\varsigma}(\mathscr{L}_t)(Z,Z') = e^{-u\mathscr{L}_t}(Z,Z') \qquad \text{if} \quad |Z - Z'| \geq c'\varsigma. \quad (2.5.74)$$

From (2.5.73), we see that for $k \in \mathbb{N}$, there is a unique holomorphic function $\widetilde{K}_{u,\varsigma,k}$ defined on a neighborhood of V_c which satisfies the same estimates as $\widetilde{K}_{u,\varsigma}$ in (2.5.73) and

$$\frac{\widetilde{K}_{u,\varsigma,k}^{(k-1)}(\lambda)}{(k-1)!} = \widetilde{K}_{u,\varsigma}(\lambda). \quad (2.5.75)$$

In particular, as in (2.5.61), we have

$$\widetilde{K}_{u,\varsigma}(\mathscr{L}_t) = \frac{1}{2i\pi}\int_\Gamma \widetilde{K}_{u,\varsigma,k}(\lambda - \mathscr{L}_t)^{-k} d\lambda. \quad (2.5.76)$$

Using (2.5.64) and proceeding as in (2.5.66)-(2.5.69), we find

$$\sup_{|\alpha|,|\alpha'|\leq m} \left(1+|Z^\perp|+|Z'^\perp|\right)^{m''} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \widetilde{K}_{u,\varsigma}(Z,Z')\right| \\ \leq C\left(1+|Z^0|+|Z'^0|\right)^{2n+2+m} \exp\left(C''c^2u - \frac{C_1}{u}\varsigma^2\right). \quad (2.5.77)$$

For $Z \neq Z'$, we set $\varsigma > 1$ such that $\left|\varsigma - \frac{1}{c'}|Z - Z'|\right| < 1$ in the previous estimate and get

$$\sup_{|\alpha|,|\alpha'|\leq m} \left(1 + |Z^{\perp}| + |Z'^{\perp}|\right)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \widetilde{K}_{u,\varsigma}(Z, Z') \right| \\ \leq C\left(1 + |Z^0| + |Z'^0|\right)^{2n+2+m} \exp\left(C'' c^2 u - \frac{C_1}{c'^2 u}|Z - Z'|^2\right). \quad (2.5.78)$$

Now, take $\delta_1 = \frac{C''c^2 + 2C_2}{C''c^2 + 4C_2}$, then from $(2.5.69)^{\delta_1} \times (2.5.78)^{1-\delta_1}$ and (2.5.74) (and from (2.5.69) if $Z = Z'$), we get (2.5.60) for $r = m' = 0$, i.e., for all $Z, Z' \in T_{x_0}B$

$$\sup_{|\alpha|,|\alpha'|\leq m} \left(1 + |Z^{\perp}| + |Z'^{\perp}|\right)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} e^{-u\mathscr{L}_t}(Z, Z') \right| \\ \leq C\left(1 + |Z^0| + |Z'^0|\right)^{2n+2+m} \exp\left(4C_2 u - \frac{C'}{u}|Z - Z'|^2\right). \quad (2.5.79)$$

We now turn to the case $r \geq 1$. By (2.5.61), we have

$$\frac{\partial^r}{\partial t^r} e^{-u\mathscr{L}_t} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_{\Gamma} e^{-\lambda} \frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_t)^{-1} d\lambda. \quad (2.5.80)$$

For $k, q \in \mathbb{N}^*$, set

$$I_{k,r} = \Big\{(\mathbf{k}, \mathbf{r}) = (k_i, r_i) \in (\mathbb{N}^*)^{j+1} \times (\mathbb{N}^*)^j \ : \ \sum_{i=0}^{j} k_i = k + j\,, \ \sum_{i=1}^{j} r_i = r\Big\}. \quad (2.5.81)$$

For $(\mathbf{k}, \mathbf{r}) \in I_{k,r}$, $\lambda \in \Gamma$, $t > 0$ set

$$A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t) = (\lambda - \mathscr{L}_t)^{-k_0} \frac{\partial^{r_1} \mathscr{L}_t}{\partial t^{r_1}} (\lambda - \mathscr{L}_t)^{-k_1} \cdots \frac{\partial^{r_j} \mathscr{L}_t}{\partial t^{r_j}} (\lambda - \mathscr{L}_t)^{-k_j}. \quad (2.5.82)$$

Then there exist $a^{\mathbf{k}}_{\mathbf{r}} \in \mathbb{R}$ such that

$$\frac{\partial^r}{\partial v^r} (\lambda - \mathscr{L}_t)^{-k} = \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,q}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t). \quad (2.5.83)$$

We claim that for any $m \in \mathbb{N}$, $k > 2(m+r+1)$ and $Q, Q' \in \cup_{\ell=1}^{m} \mathcal{D}_t^{\ell}$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $\lambda \in \Gamma$

$$\left\| Q A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t) Q' s \right\|_0 \leq C(1 + |\lambda|)^N \sum_{|\beta| \leq 2r} \|Z^{\beta} s\|_0. \quad (2.5.84)$$

Indeed, we know by (2.5.32) that $\frac{\partial^r}{\partial t^r} \mathscr{L}_t$ is a combination of

$$\Big(\frac{\partial^{r_1}}{\partial v^{r_1}} g^{ij}(tZ)\Big), \quad \Big(\frac{\partial^{r_2}}{\partial t^{r_2}} \nabla_{t,e_i}\Big), \quad \frac{\partial^{r_1}}{\partial t^{r_1}} \theta(tZ), \quad \frac{\partial^{r_1}}{\partial t^{r_1}} t \langle \tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ) \rangle, \quad (2.5.85)$$

where θ runs over the functions r^X, etc., appearing in (2.5.32).

Now, if $f = g^{ij}$ or $f = \theta$ in (2.5.85) (resp. $f = \nabla_{t,e_i}$ or $f = t\langle \tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ) \rangle$), then for $r_1 \geq 1$, $\frac{\partial^{r_1}}{\partial v^{r_1}} f(tZ)$ is a function of the type $g(tZ) Z^{\beta}$ where $|\beta| \leq r_1$ (resp. $r_1 + 1$) and $g(Z)$ and its derivatives in Z are uniformly bounded for $Z \in \mathbb{R}^{2n}$.

Let $\mathscr{F}'_t$ be the family of operators of the form

$$\mathscr{F}'_t = \{[f_{j_1} Q_{j_1}, [f_{j_2} Q_{j_2}, \ldots [f_{j_m} Q_{j_m}, \mathscr{L}_t] \ldots]]\}, \quad (2.5.86)$$

where f_{j_i} is smooth and bounded (with its derivatives) and $Q_{j_i} \in \mathcal{D}_t \cup \{Z_l\}_{l=1}^{2n-d}$.

We will now deal with the operator $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$. First, we move all the terms Z^β in the terms $g(tZ)Z^\beta$ (defined above) to the right-hand side of this operator. To do so, we use the same commutator trick as in the proof of Theorem 2.5.9, that is we perform the commutations once at a time with each Z_i (and not directly with Z^β, $|\beta| > 1$). Then we obtain that $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$ is of the form $\sum_{|\beta|\le 2r} L_{t,\beta} Q''_\beta Z^\beta$ where Q''_β is obtained from Q' and its commutation with Z^β. Next, we move all the terms ∇_{t,e_i} and $\langle \frac{1}{t}\tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ)\rangle$ in $\frac{\partial^r}{\partial t^r}\mathscr{L}_t$ to the right-hand side of the operators $L_{t,\beta}$. Then as in the proof of Theorem 2.5.9, we finally get that $QA_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$ is of the form $\sum_{|\beta|\le 2r} \mathscr{L}_{t,\beta} Z^\beta$, where $\mathscr{L}_{t,\beta}$ is a linear combination of operators of the type

$$Q(\lambda - \mathscr{L}_t)^{-k'_0} R_1 (\lambda - \mathscr{L}_t)^{-k'_1} R_2 \cdots R_{l'} (\lambda - \mathscr{L}_t)^{-k'_{l'}} Q''' Q'', \tag{2.5.87}$$

where $\sum_j k'_j = k + l'$, $R_j \in \mathscr{F}'_t$, $Q''' \in \cup_{\ell=1}^{2r} \mathcal{D}_t^\ell$ and $Q'' \in \cup_{\ell=1}^{m} \mathcal{D}_t^\ell$ is obtained from Q' and its commutation with Z^β. Since $k > 2(m + r + 1)$, we can use Proposition 2.5.9 and the arguments leading to (2.5.62) and (2.5.63) in order to split the operator in (2.5.87) into two parts:

$$\begin{aligned} Q(\lambda - \mathscr{L}_t)^{-k'_0} R_1 (\lambda - \mathscr{L}_t)^{-k'_1} R_2 \cdots R_i (\lambda - \mathscr{L}_t)^{-k''_i} \times \\ (\lambda - \mathscr{L}_t)^{-(k'_i - k''_i)} R_{i+1} \cdots R_{l'} (\lambda - \mathscr{L}_t)^{-k'_{l'}} Q''' Q'', \end{aligned} \tag{2.5.88}$$

such that the $\|\cdot\|_t^{0,0}$-norm each part is bounded by $C(1 + |\lambda|^2)^N$. This conclude the proof of (2.5.84).

By (2.5.80), (2.5.83) and (2.5.84), we get (2.5.60) for $m' = 0$ using a similar reasoning that for (2.5.79).

For $m' = 1$, observe that if $U \in TM_G$, then

$$\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} e^{-u\mathscr{L}_p} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_\Gamma e^{-\lambda} \nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} (\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{2.5.89}$$

Moreover, $\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} (\lambda - \mathscr{L}_t)^{-k}$ is a linear combination operators of the form

$$(\lambda - \mathscr{L}_t)^{-i_1} (\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} \mathscr{L}_t)(\lambda - \mathscr{L}_t)^{-i_2} \dots (\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} \mathscr{L}_t)(\lambda - \mathscr{L}_t)^{-i_\ell}, \tag{2.5.90}$$

and $\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} \mathscr{L}_t$ is a differential operator with the same structure as $\mathscr{L}_t$. In particular, $\nabla_U^{\pi_{M_G}^* \mathrm{End}(\mathcal{E})} \mathscr{L}_t$ satisfies an estimates analogous to (2.5.51). Thus, above arguments can be repeated to prove (2.5.60) for $m' = 1$. The case $m' \ge 2$ is similar. □

Remark 2.5.11. In the sequel, we will in fact only use Theorem 2.5.10 with $r = 0, 1$, but we prefer to state it in the general case.

Proposition 2.5.12. *There are constants $C > 0$ and $M \in \mathbb{N}^*$ such that for $t \in [0, t_0]$ and $\lambda \in \Gamma$,*

$$\left\| ((\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1}) s \right\|_{0,0} \le Ct(1 + |\lambda|^2)^M \sum_{|\alpha|\le 3} ||Z^\alpha s||_{0,0}. \tag{2.5.91}$$

Proof. From (2.5.29) and (2.5.39), for $t \in [0, 1]$ and $m \in \mathbb{N}^*$ we find

$$\|s\|_{t,m} \le C \sum_{|\alpha|\le m} \|Z^\alpha s\|_{0,m}. \tag{2.5.92}$$

Moreover, for s, s' with compact support, a Taylor expansion of (2.5.32) gives

$$\left|\langle(\mathscr{L}_t-\mathscr{L}_0)s,s'\rangle_{t,0}\right| \leq Ct\|s'\|_{t,1}\sum_{|\alpha|\leq 3}\|Z^\alpha s\|_{0,1}. \tag{2.5.93}$$

Thus,

$$\|(\mathscr{L}_t-\mathscr{L}_0)s\|_{t,-1} \leq Ct\sum_{|\alpha|\leq 3}\|Z^\alpha s\|_{0,1}. \tag{2.5.94}$$

Note that

$$(\lambda-\mathscr{L}_t)^{-1}-(\lambda-\mathscr{L}_0)^{-1}=(\lambda-\mathscr{L}_t)^{-1}(\mathscr{L}_t-\mathscr{L}_0)(\lambda-\mathscr{L}_0)^{-1}. \tag{2.5.95}$$

Moreover, Propositions 2.5.7, 2.5.8 and 2.5.9 still holds for $t=0$. Thus, Proposition 2.5.9, (2.5.94) and (2.5.95) yields to (2.5.91). □

Theorem 2.5.13. *There exists $C'>0$ such that for any $m, m', m''\in\mathbb{N}$ and $u_0>0$, there is $C>0$ such that for any $t\in]0,t_0]$, $u\geq u_0$ and $Z, Z'\in B_0$*

$$\begin{aligned}\sup_{|\alpha|,|\alpha'|\leq m}\left(1+|Z^\perp|+|Z'^\perp|\right)^{m''}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\left(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0}\right)(Z,Z')\right|_{\mathscr{C}^{m'}(M_G)}\\ &\leq Ct(1+|Z^0|+|Z'^0|)^{2(n+m'+1)+m}\exp\left(4C_2u-\frac{C'}{u}|Z-Z'|^2\right).\end{aligned} \tag{2.5.96}$$

Proof. Let $\mathcal{B}_q=B^{T_{x_0}B}(0,q)$. Let $\|s\|^2_{\mathcal{B}_q}=\int_{|Z|\leq q}|s|^2_{h^{\mathcal{E}_{x_0}}}dv_{TX}(Z)$, and let $J_{q,x_0}=L^2(\mathcal{B}_q,\mathcal{E}_{B,x_0})$. If A is a bounded operator on J_{q,x_0}, we denote its operator norm by $\|A\|_{\mathcal{B}_q}$. By (2.5.61) and (2.5.91), we know that there is $C'>0$ and $N, M\in\mathbb{N}$ such that for $t\in]0,1]$,

$$\begin{aligned}\left\|e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0}\right\|_{\mathcal{B}_q}&\leq\frac{1}{2\pi}\int_\Gamma|e^{-u\lambda}|\left\|(\lambda-\mathscr{L}_t)^{-1}-(\lambda-\mathscr{L}_0)^{-1}\right\|_{\mathcal{B}_q}d\lambda\\ &\leq Ct\int_\Gamma e^{-u\mathrm{Re}(\lambda)}(1+|\lambda|^2)^M(1+q)^Nd\lambda\leq C't(1+q)^N.\end{aligned} \tag{2.5.97}$$

Let $\phi\colon T_{x_0}B\to[0,1]$ be a smooth function with compact support, equal to 1 near 0 and such that $\int_{T_{x_0}B}\phi(Z)dv_{TX}(Z)=1$. Let $\nu\in]0,1]$. By the proof of Theorem 2.5.10, we see that $e^{-u\mathscr{L}_0}$ satisfies an inequality similar to (2.5.60). By Theorem 2.5.10, there exists $C>0$ such that for $|Z|,|Z'|\leq q$ and $U, U'\in\mathcal{E}_{x_0}$,

$$\begin{aligned}&\Big|\Big\langle(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0})(Z,Z')U,U'\Big\rangle\\ &\qquad-\int_{T_{x_0}B\times T_{x_0}B}\Big\langle(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0})(Z-W,Z'-W')U,U'\Big\rangle\\ &\qquad\times\frac{1}{\nu^{4n-2d}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big|\leq C\nu(1+q)^N|U||U'|.\end{aligned} \tag{2.5.98}$$

Moreover, by (2.5.97), we have

$$\begin{aligned}&\Big|\int_{T_{x_0}B\times T_{x_0}B}\Big\langle(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0})(Z-W,Z'-W')U,U'\Big\rangle\\ &\qquad\times\frac{1}{\nu^{4n-2d}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big|\leq\frac{Ct}{\nu^{2n-d}}(1+q)^N|U||U'|.\end{aligned} \tag{2.5.99}$$

Hence, taking $\nu = t^{1/(2n-d+1)}$ we find that there is $C > 0$ and $K \in \mathbb{N}$ such that for any $t \in]0, t_0]$, $Z, Z' \in B^{B_0}(0, q)$,

$$\left|(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0})(Z, Z')\right| \leq Ct^{1/(2n-d+1)}(1+q)^K. \tag{2.5.100}$$

In particular, we have

$$e^{-u\mathscr{L}_t}|_{t=0} = e^{-u\mathscr{L}_0}. \tag{2.5.101}$$

From Theorem 2.5.10, (2.5.101) and the formula

$$G(t) - G(0) = \int_0^t G'(s)ds, \tag{2.5.102}$$

we get (2.5.96). □

Remark 2.5.14. As we have estimates on every derivatives of $e^{-u\mathscr{L}_t}(Z, Z')$, we can in fact use the same method as in Theorem 2.5.13 to get an asymptotic expansion at every order of $e^{-u\mathscr{L}_t}(Z, Z')$.

2.5.3 Computation of the limiting heat kernel

In this section, we will evaluate the limiting heat kernel $e^{-u\mathscr{L}_0}((0, Z^\perp), (0, Z^\perp))$ for $(0, Z^\perp) \in T_{x_0}B$ and thus obtain Theorem 2.1.9.

Recall that we have the following splitting of vector bundle over P, which is orthogonal for both b^L and g^{TM} (see (2.1.24) and (2.3.19)):

$$TU = T^HP \oplus TY \oplus JTY. \tag{2.5.103}$$

Note also that by (2.1.11) and (2.1.25), we have

$$b^L(\cdot\,, \cdot) = \langle (-J\mathbf{J})\cdot\,, \cdot \rangle, \tag{2.5.104}$$

and thus $-J\mathbf{J}$ preserves both TY and JTY on P. In particular, on P, $\mathbf{J}$ intertwines TY and JTY, and is invertible on $TY \oplus JTY$ because g^{TM} and b^L are definite positive on this bundle. Thus,

$$\mathbf{J}^2TY = TY\,, \quad \mathbf{J}TY = JTY\,, \quad \mathbf{J}T^HP = JT^HP = T^HP. \tag{2.5.105}$$

Thus, $\mathbf{J}$ induces naturally $\mathbf{J}_G \in \operatorname{End}(TM_G)$, and we see with (2.5.3) that $(\mathbf{J}TY)_B|_{M_G}$ is the orthogonal complement of TM_G in TB. We will identify the normal bundle N_G of M_G in B with $(\mathbf{J}TY)_B|_{M_G}$. From this fact and (2.5.105), we know that for $U, V \in T_{x_0}B$,

$$\omega(U^H, V^H) = \omega_G(P^{TM_G}U, P^{TM_G}V). \tag{2.5.106}$$

From the above discussion, we can diagonalize $\mathbf{J}$ on $(T^HP)^{(1,0)}$ and $(TY \oplus JTY)^{(1,0)}$, and we thus can get orthonormal basis $\{w_j^0\}_{j=1}^{n-d}$ and $\{e_i^\perp\}_{i=1}^d$ of $T_{x_0}^{(1,0)}M_G$ and $N_{G,x_0} = (\mathbf{J}TY)_{B,x_0} \subset TB$ respectively such that in these basis

$$\begin{cases} \mathbf{J}|_{T_{x_0}^{(1,0)}M_G} = \dfrac{\sqrt{-1}}{2\pi}\operatorname{diag}(a_1^0, \ldots, a_{n-d}^0), \\ \mathbf{J}^2|_{N_{G,x_0}} = -\dfrac{1}{4\pi^2}\operatorname{diag}(a_1^{\perp,2}, \ldots, a_d^{\perp,2}), \end{cases} \tag{2.5.107}$$

with $a_j^0 \in \mathbb{R}$ and $a_j^\perp \in \mathbb{R}^*$. Let $\{w^{0,j}\}_{j=1}^{n-d}$ and $\{e^{\perp,i}\}_{i=1}^{d}$ be their dual basis. We also set

$$e_{2j-1}^0 = \frac{1}{\sqrt{2}}(w_j^0 + \overline{w}_j^0) \quad \text{and} \quad e_{2j}^0 = \frac{\sqrt{-1}}{\sqrt{2}}(w_j^0 - \overline{w}_j^0). \tag{2.5.108}$$

Then $\{e_i^0\}_{i=1}^{2n-2d}$ is an orthonormal basis of $T_{x_0}M_G$.

From now on, we will use the coordinates in Section 2.5.1 induced by the above basis as in (2.4.1).

We denote by $Z^0 = (Z_1^0, \dots, Z_{2n-2d}^0)$ and $Z^\perp = (Z_1^\perp, \dots, Z_d^\perp)$ the elements in $T_{x_0}M_G$ and N_{G,x_0}. Then $Z \in T_{x_0}B$ can be decomposed as $Z = (Z^0, Z^\perp)$. We will also use the complex coordinates $z^0 = (z_1^0, \dots, z_{n-d}^0)$, so that

$$\begin{aligned} &Z^0 = z^0 + \overline{z}^0, \\ &w_j^0 = \sqrt{2}\frac{\partial}{\partial z_j^0}, \quad \overline{w}_j^0 = \sqrt{2}\frac{\partial}{\partial \overline{z}_j^0}, \\ &e_{2j-1}^0 = \frac{\partial}{\partial z_j^0} + \frac{\partial}{\partial \overline{z}_j^0}, \quad e_{2j}^0 = \sqrt{-1}(\frac{\partial}{\partial z_j^0} - \frac{\partial}{\partial \overline{z}_j^0}). \end{aligned} \tag{2.5.109}$$

When we consider z^0 or $\overline{z}^0$ as vector fields, we identify them with $\sum_j z_j^0 \frac{\partial}{\partial z_j^0}$ and $\sum_j \overline{z}_j^0 \frac{\partial}{\partial \overline{z}_j^0}$. Note that

$$\Big|\frac{\partial}{\partial z_j^0}\Big|^2 = \Big|\frac{\partial}{\partial \overline{z}_j^0}\Big|^2 = \frac{1}{2} \quad \text{and} \quad |z^0|^2 = |\overline{z}^0|^2 = \frac{1}{2}|Z^0|^2. \tag{2.5.110}$$

Set

$$\mathscr{L} = -\sum_{i=1}^{2n-2d} (\nabla_{0,e_i^0})^2 - \sum_{j=1}^{n-d} a_j^0, \tag{2.5.111}$$

and recall that

$$\mathscr{L}^\perp = -\sum_{i=1}^{d} \Big((\nabla_{e_i^\perp})^2 - |a_i^\perp Z_i^\perp|^2\Big) - \sum_{j=1}^{d} a_j^\perp. \tag{2.5.112}$$

As in [51, (3.13)], we can show using (2.3.15), (2.5.27) and (2.5.106), that

$$\begin{aligned} &R_{x_0}^{L_B}(U,V) = -2\pi\sqrt{-1}\langle \mathbf{J}P^{TM_G}U, P^{TM_G}V\rangle, \\ &\mathscr{L}_0 = \mathscr{L} + \mathscr{L}^\perp - 2\omega_d(x_0), \end{aligned} \tag{2.5.113}$$

Thus,

$$e^{-u\mathscr{L}_0}(Z,Z') = e^{-u\mathscr{L}}(Z^0, Z'^0) e^{-u\mathscr{L}^\perp}(Z^\perp, Z'^\perp) e^{2u\omega_{G,d}(x_0)}. \tag{2.5.114}$$

Moreover, using (2.5.107), (2.5.111), (2.5.113) and the formula for the heat kernel of a harmonic oscillator (see [46, (E.2.4), (E.2.5)] for instance), we find with the convention of Theorem 2.1.9

$$e^{-u\mathscr{L}}(0,0) = \frac{1}{(2\pi)^{n-d}} \frac{\det(\dot{R}_x^{L_G})}{\det\big(1 - \exp(-2u\dot{R}_x^{L_G})\big)}. \tag{2.5.115}$$

We can now prove Theorem 2.1.9. We fix $u > 0$.

Let $s \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{x_0})$. Then by (2.5.26) and (2.5.27)

$$\begin{aligned} e^{-u\mathscr{L}_t}s(Z) &= S_t^{-1}\kappa^{1/2} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}} \kappa^{-1/2} S_t(Z) \\ &= \kappa(tZ) \int_{\mathbb{R}^{2n-d}} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,Z')(S_t s)(Z')\kappa^{1/2}(Z') dv_{TX}(Z') \\ &= p^{-n+d/2}\kappa(tZ) \int_{\mathbb{R}^{2n-d}} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,tZ'')s(Z'')\kappa^{1/2}(tZ'') dv_{TX}(Z''), \end{aligned} \tag{2.5.116}$$

which yields to

$$e^{-u\mathscr{L}_t}(Z,Z') = p^{-n+d/2} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,tZ')\kappa^{1/2}(tZ)\kappa^{-1/2}(tZ'). \tag{2.5.117}$$

On the other hand, for $s \in \mathscr{C}_c^\infty(B_0, (\mathbb{E}_{0,p})_{B_0})$ and $v \in M_0$,

$$\begin{aligned}\left(e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}} s\right)(\pi(v)) &= \left(\Phi e^{-\frac{u}{p}D_p^{M_0,2}}\Phi^{-1}s\right)(\pi(v)) \\ &= h(v)\int_{M_0} e^{-\frac{u}{p}D_p^{M_0,2}}(v,v')h^{-1}(v')s(v')dv_{M_0}(v') \\ &= h(v)\int_{B_0} e^{-\frac{u}{p}D_p^{M_0,2}}(v,y')h(y')s(y')dv_{B_0}(y'),\end{aligned} \tag{2.5.118}$$

thus we find

$$h(v)h(v')(P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G)(v,v') = e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(\pi(v),\pi(v')). \tag{2.5.119}$$

Let $v = (g,Z) \in U \simeq G \times B^{T_{x_0}B}(0,\varepsilon)$. We suppose that in the decomposition $Z = Z^0 + Z^\perp$, we have $Z^0 = 0$. Then from Corollary 2.5.2, Theorem 2.5.13, (2.5.117), and (2.5.119), we find that for any $m, m' \in \mathbb{N}$, there exists $C > 0$ (independent of $Z^\perp$) such

$$\begin{aligned}\Big| p^{-n+d/2}h(v)h(v)(P_G e^{-\frac{u}{p}D_p^2}P_G)(v,v) - \kappa^{-1}(Z^\perp)e^{-u\mathscr{L}_0}(\sqrt{p}Z^\perp,\sqrt{p}Z^\perp)\Big|_{\mathscr{C}^{m'}(M_G)} \\ \leq Cp^{-1/2}\big(1+\sqrt{p}|Z^\perp|\big)^{-m}.\end{aligned} \tag{2.5.120}$$

Now, for $v \in U$, we write as in the Introduction of this chapter $v = (y, Z^\perp)$ with $y \in P$ and $Z^\perp \in N_{P/U,y}$. Let $x = \pi(y) \in M_G$. Then we do the procedure of Sections 2.5.1 and 2.5.3 with $x_0 = x$ and $y_0 = y$. Then Theorem 2.1.9 follows from (2.5.114), (2.5.115) and (2.5.120) applied to $Z = (0, Z^\perp) \in T_{x_0}B = T_{x_0}M_G \oplus N_{G,x_0}$.

2.6 Proof of the inequalities

In this Section, we prove our main results: Theorems 2.1.3 and 2.1.5. In Section 2.6.1 we prove Theorem 2.1.7 and, as a consequence, we obtain the G-invariant holomorphic Morse inequalities in the case of a free G-action on P. Then, we explain in Section 2.6.2 how to modify the arguments in Sections 2.5 and 2.6.1 to get our inequalities under Assumption 2.1.1 in full generality. Finally, in Section 2.6.3, we apply Theorem 2.1.5 to get estimates on the other isotypic components of the cohomology $H^\bullet(M, L^p \otimes E)$.

2.6.1 Proof of Theorem 2.1.3 when G acts freely on P

We assume in this Section that G acts freely on P and $\overline{U}$. We keep here the notations of Sections 2.5.

In this section, we will first prove Theorem 2.1.7, and then show how to use it in conjunction with the convergence of the heat kernel of the rescaled operator to get Theorem 2.1.3. The method is inspired by [6] (see also [46, Sect. 1.7]).

For $0 \leq q \leq n$, set

$$b_q^{p,G} = \dim H^q(M, L^p \otimes E)^G. \tag{2.6.1}$$

By Hodge theory, there is a G-equivariant isomorphism $H^\bullet(M, L^p \otimes E) \simeq \ker D_p^2$, and in particular we get for the invariant part:

$$H^\bullet(M, L^p \otimes E)^G \simeq (\ker D_p^2)^G \quad \text{and} \quad b_q^{p,G} = \dim(\ker D_p^2)^G. \tag{2.6.2}$$

We begin by proving Theorem 2.1.7.

Proof of Theorem 2.1.7. If λ is an eigenvalue of D_p^2 acting on $\Omega^{0,j}(M, L^p \otimes E)^G$, we denote by F_j^λ the corresponding finite-dimensional eigenspace. As $\bar\partial^{L^p\otimes E}$ and $\bar\partial^{L^p\otimes E,*}$ commute with D_p^2, we deduce that

$$\bar\partial^{L^p\otimes E}(F_j^\lambda) \subset F_{j+1}^\lambda \qquad \text{and} \qquad \bar\partial^{L^p\otimes E,*}(F_j^\lambda) \subset F_{j-1}^\lambda. \tag{2.6.3}$$

As a consequence, we have a complexe

$$0 \longrightarrow F_0^\lambda \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} F_1^\lambda \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} \cdots \overset{\bar\partial^{L^p\otimes E}}{\longrightarrow} F_n^\lambda \longrightarrow 0. \tag{2.6.4}$$

If $\lambda = 0$, we have $F_j^0 \simeq H^j(M, L^p \otimes E)^G$ by (2.6.2). If $\lambda > 0$, then the complex (2.6.4) is exact. Indeed, if $\bar\partial^{L^p\otimes E} s = 0$ and $s \in F_j^\lambda$, then

$$s = \lambda^{-1} D_p^2 s = \lambda^{-1} \bar\partial^{L^p\otimes E} \bar\partial^{L^p\otimes E,*} s \in \operatorname{Im}(\bar\partial^{L^p\otimes E}). \tag{2.6.5}$$

In particular, we get for $\lambda > 0$

$$\sum_{j=0}^{q} (-1)^{q-j} \dim F_j^\lambda = \dim \left(\bar\partial^{L^p\otimes E}(F_q^\lambda)\right) \geq 0, \tag{2.6.6}$$

with equality if $q = n$.

Now,

$$\operatorname{Tr}_j[P_G e^{-\frac{u}{p} D_p^2} P_G] = b_j^{p,G} + \sum_{\lambda > 0} e^{-\frac{u}{p}\lambda} \dim F_j^\lambda. \tag{2.6.7}$$

Thus, (2.6.6) and (2.6.7) entail (2.1.19).

Note that this proof does not depend on the metric we chose on TM, so we get (2.1.19) in general. ☐

We denote by $\operatorname{Tr}_{\Lambda^{0,q}}$ the trace on $\Lambda^{0,q}(T^*M) \otimes L^p \otimes E$ or on $\Lambda^{0,q}(T^*M)$. We know that

$$\operatorname{Tr}_q[P_G e^{-\frac{u}{p} D_p^2} P_G] = \int_M \operatorname{Tr}_{\Lambda^{0,q}} \left[(P_G e^{-\frac{u}{p} D_p^2} P_G)(v, v)\right] dv_M(v). \tag{2.6.8}$$

With Theorem 2.1.8 and (2.5.23), we in fact have

$$\operatorname{Tr}_q[P_G e^{-\frac{u}{p} D_p^2} P_G] = \int_U \operatorname{Tr}_{\Lambda^{0,q}} \left[(P_G e^{-\frac{u}{p} D_p^2} P_G)(v, v)\right] dv_M(v) + O(p^{-\infty}). \tag{2.6.9}$$

By Theorems 2.1.7 and 2.1.9, (2.6.9), and using the change of variable $Z^\perp \leftrightarrow \sqrt{p} Z^\perp$, we deduce that for every $u > 0$,

$$p^{-n+d} \sum_{j=0}^{q} (-1)^{q-j} b_j^{p,G} \leq$$
$$\frac{\operatorname{rk}(E)}{(2\pi)^{n-d}} \int_{x \in M_G,\, |Z^\perp| \leq \sqrt{p}\varepsilon} \frac{\det(\dot{R}_x^{L_G}) \sum_{j=0}^{q} (-1)^{q-j} \operatorname{Tr}_{\Lambda^{0,j}}[e^{2u\omega_d(x)}]}{\det\left(1 - \exp(-2u\dot{R}_x^{L_G})\right)} e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z^\perp) dv_{TB}(x, Z^\perp)$$
$$+ o(1). \tag{2.6.10}$$

For $u > 0$, set

$$f(u) = \frac{1}{\tanh(2u)} - \frac{1}{\sinh(2u)}. \tag{2.6.11}$$

Then there is $c > 0$ such that for $u > 1$, $f(u) > c$, and $f(u) \xrightarrow[u\to\pm\infty]{} \pm 1$. By (2.1.28) and Mehler's formula (see [46, Thm. E.1.4] for instance), we know that

$$e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp}, Z^{\perp}) = \prod_{i=1}^{d} \sqrt{\frac{a_i^{\perp}}{\pi(1-e^{-4ua_i^{\perp}})}} \exp\left\{-a_i^{\perp} f(ua_i^{\perp}) Z_i^{\perp,2}\right\} \tag{2.6.12}$$

Thus, as $a_i^{\perp} f(ua_i^{\perp}) > 0$,

$$\begin{aligned}\int_{|Z^{\perp}|\leq\sqrt{p}\varepsilon} e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp}, Z^{\perp}) dv_{N_{G,x}}(Z^{\perp}) &= \int_{\mathbb{R}^d} e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp}, Z^{\perp}) dv_{N_{G,x}}(Z^{\perp}) + O(p^{-\infty}) \\ &= \prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}} + O(p^{-\infty}).\end{aligned} \tag{2.6.13}$$

Let $\{w_j^0\}$ be a local orthonormal frame of $T^{(1,0)}M_G$ such that $\dot{R}^{L_G} w_j^0 = a_j^0 w_j^0$ (see (2.5.107)). Its dual frame is denoted by $\{w^{0,j}\}$. Then

$$\omega_{G,d} = -\sum_{j=1}^{n-d} a_j^0\, \overline{w}^{0,j} \wedge i_{\overline{w}_j^0}. \tag{2.6.14}$$

We again denote by w_j^0 the horizontal lift of w_j^0 in $T^H P$. In the same way, let $\{w_j^{\perp}\}$ be a local orthonormal frame of $(TY \oplus JTY)^{(1,0)}$ such that $\dot{R}^L w_j^{\perp} = a_j^{\perp} w_j^0$. Its dual frame is denoted by $\{w^{\perp,j}\}$. Then

$$\omega_d = -\sum_{j=1}^{n-d} a_j^0\, \overline{w}^{0,j} \wedge i_{\overline{w}_j^0} - \sum_{j=1}^{d} a_j^{\perp}\, \overline{w}^{\perp,j} \wedge i_{\overline{w}_j^{\perp}}. \tag{2.6.15}$$

Thus, writing $\{w_j\} = \{w_j^0, w_j^{\perp}\}$ and $\{a_j\} = \{a_j^0, a_j^{\perp}\}$, we get

$$e^{2u\omega_d} = 1 + \sum_j (e^{-2ua_j} - 1)\overline{w}^j \wedge i_{\overline{w}_j}, \tag{2.6.16}$$

and

$$\operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d}] = \sum_{j_1<\cdots<j_q} \exp\Big(-2u\sum_{k=1}^{q} a_{j_k}\Big). \tag{2.6.17}$$

In particular, there exist $C > 0$ such that for $x \in M_G$, $u > 1$ and $0 \leq q \leq n$,

$$\left|\frac{\det(\dot{R}_x^{L_G}) \operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(x)}]}{\det\left(1-\exp(-2u\dot{R}_x^{L_G})\right)} \prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}}\right| \leq C. \tag{2.6.18}$$

On the other hand the signature of b^L on JTY is the same as on TY (i.e., $(r, d-r)$), so by Lemma 2.3.3 and (2.3.15), (2.3.19) and (2.1.10) we have for $0 \leq q \leq n$

$$\pi(P \cap M(q)) = M_G(q-r), \tag{2.6.19}$$

where $M(\leq q)$ is define in an analogue way as $M_G(\leq q)$ in the introduction. Thus, by (2.6.17) and (2.6.19),

$$\begin{aligned}\lim_{u\to+\infty} \frac{\det(\dot{R}_x^{L_G}) \operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(x)}]}{\det\left(1-\exp(-2u\dot{R}_x^{L_G})\right)} &\prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}} \\ &= \mathbf{1}_{M_G(q-r)}(x)(-1)^{q-r} \det(\dot{R}^{L_G}),\end{aligned} \tag{2.6.20}$$

where the function $\mathbf{1}_A$ takes the value 1 on A and 0 elsewhere.

Using (2.6.10), (2.6.13), (2.6.18), (2.6.20) and dominated convergence as $u \to +\infty$, we find

$$\begin{aligned}\limsup_{p\to+\infty} p^{-n+d}\sum_{j=0}^{q}(-1)^{q-j}b_j^{p,G} &\leq \frac{\mathrm{rk}(E)}{(2\pi)^{n-d}}\prod_{i=1}^{d}\sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}}\times \\ &\int_{M_G}\frac{\det(\dot{R}_x^{L_G})\sum_{j=0}^{q}(-1)^{q-j}\operatorname{Tr}_{\Lambda^{0,j}}[e^{2u\omega_{G,d}(x)}]}{\det\left(1-\exp(-2u\dot{R}_x^{L_G})\right)}dv_{M_G}(x) \\ &\leq (-1)^{q-r}\int_{M_G(\leq q-r)}\det\Big(\frac{\dot{R}_x^{L_G}}{2\pi}\Big)dv_{M_G}(x).\end{aligned} \tag{2.6.21}$$

Finally, note that

$$\det\Big(\frac{\dot{R}_x^{L_G}}{2\pi}\Big)dv_{M_G}(x) = \Big(\frac{\sqrt{-1}}{2\pi}R^{L_G}\Big)^{n-d}/(n-d)! = \frac{\omega_G^{n-d}}{(n-d)!}. \tag{2.6.22}$$

Then (2.6.21) and (2.6.22) entail Theorem 2.1.3.

2.6.2 The case of a locally free action

In this section, we prove Theorem 2.1.3 under Assumption 2.1.1. In particular, the action of G on P and $\overline{U}$ is only locally free, and thus M_G and B are orbifolds. The proof relies on a similar method as the case of a free G-action, but the main difference is that we need to work off-diagonal to get uniform estimates near the orbifold singularities. We explain below how to adapt the arguments in Sections 2.5 and 2.6.1 to get the general result.

Recall that $G^0 = \{g \in G \,:\, g\cdot x = x \text{ for any } x \in M\}$. Then G^0 is a finite normal subgroup of G and the quotient G/G^0 acts effectively on M.

It is a well-known fact that if $\phi\colon (M, g^{TM}) \to (M, g^{TM})$ is an isometry and $x \in M$ is a point such that $\phi(x) = x$ and $d\phi_x = \mathrm{Id}_{T_xM}$ then $\phi = \mathrm{Id}_M$. In particular, suppose that $g \in G$ satisfies $g|_P = \mathrm{Id}_P$. Then we have for $x \in P$: $gx = x$, $dg_x|_{T_xP} = \mathrm{Id}_{T_xP}$ and g preserves J so $dg_x|_{JT_xP} = \mathrm{Id}_{JT_xP}$. As $TP + JTP = TM$, we deduce that g acts as the identity on M. Thus,

$$G^0 = \{g \in G \,:\, g\cdot x = x \text{ for any } x \in P\}. \tag{2.6.23}$$

Recall that the function h defined in (2.2.9) is smooth only on the regular part of B and we have denoted by $\widehat{h}$ its smooth extension from the regular part of B to B.

First, we need to modify Section 2.5.1 as follows.

Recall that TM is endowed with a metric g^{TM} satisfying (2.1.24). We identify the normal bundle N of P in U to the orthogonal complement of TP. By (2.1.23) and (2.1.24), this means that N is identified with JTY. By (2.1.23) and (2.3.17), we have in particular $T^HU = T^HP \oplus N$.

Let g^{TY}, g^{T^HU} be the restriction of g^{TM} on TY, T^HU. Let g^{TB} (resp. g^{TM_G}) be the metric on TB (resp. TM_G) induced by g^{T^HU} (resp. g^{T^HP}).

Here, unlike in Section 2.5, we will not work on the quotient B but directly on M. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}). Let P^N and P^{T^HP} be the orthogonal projections from $T^HU|_P$ to N and T^HP respectively. Set

$$\begin{aligned}\nabla^{T^HU} &= \pi^*\nabla^{TB}, & \nabla^N &= P^N(\nabla^{T^HU}|_P)P^N, \\ \nabla^{TP} &= P^{TP}(\nabla^{T^HU}|_P)P^{TP}, & {}^0\nabla^{T^HU} &= \nabla^N \oplus \nabla^{T^HP}.\end{aligned} \tag{2.6.24}$$

Fix $y_0 \in P$. For $V \in T^HU$ (resp. T^HP), we define $t \mapsto x_t = \exp_{y_0}^{T^HU}(tV) \in U$ (resp. $\exp_{y_0}^{T^HP}(tV) \in P$) the curve such that $x_0 = y_0$, $\dot{x}_0 = V$, $\dot{x} \in T^HU$ and $\nabla_{\dot{x}}^{T^HU}\dot{x} = 0$ (resp. $\dot{x} \in T^HP$ and $\nabla_{\dot{x}}^{T^HP}\dot{x} = 0$). For $W \in T^HP$ small and $V \in N_{y_0}$, let $\tau_W V$ be the parallel transport of V with respect to ∇^N along to curve $t \in [0,1] \mapsto \exp_{y_0}^{T^HP}(tW)$.

As in Section 2.5.1, we identify $B^{T^H_{y_0}U}(0,\varepsilon)$ to a subset of U as follows: for $Z \in B^{T^H_{y_0}U}(0,\varepsilon)$, we decompose Z as $Z = Z^0 + Z^\perp$ with $Z^0 \in T^H_{y_0}P$ and $Z^\perp \in N_{y_0}$, and then we identify Z with $\exp^{T^HU}_{\exp_{y_0}^{T^HP}(Z^0)}(\tau_{Z^0}Z^\perp)$.

Moreover, if $G_{y_0} = \{g \in G \;:\; gy_0 = y_0\}$ is the stabilizer of y_0 and $g \in G_{y_0}$, we can decompose $T^H_{y_0}P$ as

$$T^H_{y_0}P = (T^H_{y_0}P)^g \oplus \mathcal{N}_{y_0,g}, \tag{2.6.25}$$

where $(T^H_{y_0}P)^g$ is the fixed point-set of g in $T^H_{y_0}P = T_{y_0}P \cap JT_{y_0}P$ and $\mathcal{N}_{y_0,g}$ is its orthogonal complement. Hence we get, for each $g \in G_{y_0}$, a decomposition of the coordinate Z^0 as $Z^0 = Z^0_{1,g} + Z^0_{2,g}$ with $Z^0_{1,g} \in (T^H_{y_0}P)^g$ and $Z^0_{2,g} \in \mathcal{N}_{y_0,g}$. Note that $\mathrm{rk}(\mathcal{N}_{y_0,g}) = 0$ if and only if $g \in G^0$.

Observe that $U \simeq G \cdot B^{T^H_{y_0}U}(0,\varepsilon) = G \times_{G_{y_0}} B^{T^H_{y_0}U}(0,\varepsilon)$ is a G-neighborhood of the orbit $G \cdot y_0$ and $(B^{T^H_{y_0}U}(0,\varepsilon), G_{y_0})$ gives local chart on B.

As the constructions in Section 2.5.1 are G_{y_0}-equivariant, we can extend in the same way the geometric objects from $G \times_{G_{y_0}} B^{T^H_{y_0}U}(0,\varepsilon)$ to

$$M_0 := G \times_{G_{y_0}} \mathbb{R}^{2n-d}, \tag{2.6.26}$$

where $\mathbb{R}^{2n-d} \simeq T^H_{y_0}U$. Note that Lemma 2.5.1 and Corollary 2.5.2 still hold, because do not work on the quotient to get them: we only use finite propagation speed of the wave equation on M.

Set

$$\begin{aligned} &B_0 = M_0/G = \mathbb{R}^{2n-d}/G_{y_0}, \\ &\widehat{M}_0 = G \times \mathbb{R}^{2n-d}, \qquad\qquad \widehat{B}_0 = \widehat{M}_0/G = \mathbb{R}^{2n-d}. \end{aligned} \tag{2.6.27}$$

Then we have a covering $\widehat{M}_0 \to M_0$ (resp. $\widehat{B}_0 \to B_0$) which gives a (global) orbifold chart on M_0 (resp. B_0). We can then extend the geometric objects from M_0 to $\widehat{M}_0$. We will add a hat to denote the corresponding objects on $\widehat{B}_0$ or $\widehat{M}_0$. In particular, we have a Dirac operator $D_p^{\widehat{M}_0}$ on $\widehat{M}_0$ corresponding to $D_p^{M_0}$ in Section 2.5.1.

Let $\widehat{\pi}_G \colon G \times \mathbb{R}^{2n-d} \to \mathbb{R}^{2n-d}$ be the projection on the second factor. As in (2.2.10), we define

$$\widehat{\Phi} = \widehat{h}\widehat{\pi}_G \colon \mathscr{C}^\infty(G \times \mathbb{R}^{2n-d}, \mathbb{E}_{0,p})^G \to \mathscr{C}^\infty(\mathbb{R}^{2n-d}, (\mathbb{E}_{0,p})_{\widehat{B}_0}). \tag{2.6.28}$$

We also denote by $\widehat{\Phi}$ the map induced from $\mathscr{C}^\infty(M_0, \mathbb{E}_{0,p})^G$ to $\mathscr{C}^\infty(B_0, (\mathbb{E}_{0,p})_{B_0})$.

Let g^{TM_0} be defined as in (2.5.18) and let $g^{T^HM_0}$ be the metric on $\mathbb{R}^{2n-d}$ induced by g^{TM_0}, with corresponding Riemannian volume on $(\mathbb{R}^{2n-d}, g^{T^HM_0})$ denoted by $dv_{T^HM_0}$.

Let $e^{-u\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}}$ be the heat kernel of the operator $\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}$ on $\widehat{B}_0$ and $e^{-u\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}}(Z,Z')$ $(Z, Z' \in \widehat{B}_0)$ be its smooth kernel with respect to $dv_{T^HM_0}(Z')$. Concerning heat kernels on orbifolds, we refer the reader to [42, Sect. 2.1]. Then we have for $v = [g,Z]$ and $v' = [g', Z']$ in M_0,

$$\begin{aligned} \widehat{h}(v)\widehat{h}(v')(P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G)(v,v') &= e^{-\frac{u}{p}\widehat{\Phi}D_p^{M_0,2}\widehat{\Phi}^{-1}}(\pi(v),\pi(v')) \\ &= \frac{1}{|G^0|}\sum_{g \in G_{y_0}} (g,1) \cdot e^{-\frac{u}{p}\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z, Z'), \end{aligned} \tag{2.6.29}$$

where $|G^0|$ is the cardinal of G^0. Indeed, the first equality in (2.6.29) is analogous to (2.5.119), and the second from a similar computation as in [26, (5.19)] or [46, (5.4.17)].

Note that our trivialization of the restriction of L (resp. E) on $B^{T^H_{y_0}U}(0,\varepsilon)$ is not G_{y_0}-invariant, except if G_{y_0} acts trivially on L_{y_0} (resp. E_{y_0}). More precisely, let $\widehat{M}_{G,0} = \mathbb{R}^{2n-2d}\times\{0\}\subset\widehat{B}_0$ and for $g\in G_{y_0}$, let $\widehat{M}^g_{G,0}$ be the fixed point-set of g in $\widehat{M}_{G,0}$. Then the action of g on $L|_{\widehat{M}^g_{G,0}}$ is the multiplication by $e^{i\theta_g}$ and θ_g is locally constant on $\widehat{M}^g_{G,0}$. Likewise, the action of g on $E|_{\widehat{M}^g_{G,0}}$ is given by $g_E\in\mathscr{C}^\infty(\widehat{M}^g_{G,0},\mathrm{End}(E))$ which is parallel with respect to ∇^E.

Now, as we work on $\widehat{B}_0$ and $\widehat{M}_0$, we can apply the results of Sections 2.5.1-2.5.3 to the operator $\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}$. We will use the same notation as in these sections, and add a subscript to indicate the base-point (e.g., κ_x, $\mathscr{L}_{0,x}$, ...). By Theorem 2.5.13 and (2.5.117), we obtain for $g\in G_{y_0}$ and $u>0$ fixed

$$\begin{aligned}\Big|p^{-n+d/2}e^{-\frac{u}{p}\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z,Z)\\ -\kappa^{-1}_{Z_{1,g}}(Z^\perp)e^{-u\mathscr{L}_{0,Z_{1,g}}}\big(\sqrt{p}g^{-1}(Z_{2,g}+Z^\perp),\sqrt{p}(Z_{2,g}+Z^\perp)\big)\Big|_{\mathscr{C}^{m'}(M_G)}\\ \le Cp^{-1/2}\big(1+\sqrt{p}|Z_{2,g}|\big)^N\big(1+\sqrt{p}|Z^\perp|\big)^{-m}\exp\big(-Cp\inf_{h\in G_{y_0}}|h^{-1}Z-Z|^2\big).\end{aligned}\tag{2.6.30}$$

On the other hand, note that there is $\rho>0$ such that for $g\in G_{y_0}$, $|g^{-1}Z-Z|^2\ge\rho|Z_{2,g}|^2$, so

$$\begin{aligned}\Big|p^{-n+d/2}e^{-\frac{u}{p}\widehat{\Phi}D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z,Z)\\ -\kappa^{-1}_{Z_{1,g}}(Z^\perp)e^{-u\mathscr{L}_{0,Z_{1,g}}}\big(\sqrt{p}g^{-1}(Z_{2,g}+Z^\perp),\sqrt{p}(Z_{2,g}+Z^\perp)\big)\Big|_{\mathscr{C}^{m'}(M_G)}\\ \le Cp^{-1/2}\big(1+\sqrt{p}|Z^\perp|\big)^{-m}\exp\big(-C'p|Z_{2,g}|^2\big).\end{aligned}\tag{2.6.31}$$

We can now prove Theorem 2.1.5. First, observe that Theorem 2.1.7 is still true here because we work on M to prove it in Section 2.6.1. Thus, we can use a similar approach to prove Theorem 2.1.5 as in Section 2.6.1.

Note that the estimate (2.6.9) still holds. Consider now a G-invariant function $\psi\in\mathscr{C}^\infty(M)$ such that the induced function (again denoted by ψ) on B is compactly supported in a small neighborhood of $x_0\in M_G$.

Similarly to (2.5.26), we denote by dv_{T^HU} the Riemannian volume of $(T^H_{y_0}U,g^{T^H_{y_0}U})$. Then, as in (2.6.10), (2.6.31) and dominated convergence imply that

$$\begin{aligned}p^{-n+d}\int_U\psi(v)\operatorname{Tr}_q\Big[\big(P_Ge^{-\frac{u}{p}D_p^{M_0,2}}P_G\big)(v,v)\Big]dv_M(v)=\\ \frac{1}{|G_{y_0}/G^0||G^0|}\sum_{g\in G_{y_0}}\frac{p^{-\mathrm{rk}(N_{y_0,g})/2}}{(2\pi)^{n-d}}\int_{A(p,\varepsilon)}\psi\Big(Z_{1,g}+\frac{Z_{2,g}}{\sqrt{p}}\Big)\operatorname{Tr}_q\Bigg[(g,1)\cdot\frac{\det(\dot{R}^{L_G}_{Z_{1,g}})e^{2u\omega_d(Z_{1,g})}}{\det\big(1-\exp(-2u\dot{R}^{L_G}_{Z_{1,g}})\big)}\\ \times e^{-u\mathscr{L}^\perp_{Z_{1,g}}}\big(g^{-1}(Z_{2,g}+Z^\perp),Z_{2,g}+Z^\perp\big)\otimes\mathrm{Id}_E\Bigg]dv_{T^HU}(Z)+o(1),\end{aligned}\tag{2.6.32}$$

where $A(p,\varepsilon)=\{|Z_{1,g}|\le\varepsilon,|Z_{2,g}|\le\varepsilon\sqrt{p},|Z^\perp|\le\varepsilon\sqrt{p}\}$. In particular, in (2.6.32), every term involving a g such that $\mathrm{rk}(N_{y_0,g})>0$, i.e., $g\notin G^0$, disappears when we look at the leading term in p.

Thus, we now consider $g \in G^0$. The action of g on M and $\Lambda^{0,\bullet}(T^*M)$ is trivial, so we have

$$\begin{aligned}\operatorname{Tr}_q \Big[(g,1) \cdot \frac{\det(\dot{R}^{L_G}_{Z^0}) e^{2u\omega_d(Z^0)}}{\det\left(1-\exp(-2u\dot{R}^{L_G}_{Z^0})\right)} \times e^{-u\mathscr{L}^{\perp}_{Z^0}}(g^{-1}Z^{\perp}, Z^{\perp}) \otimes \mathrm{Id}_E \Big] \\ = e^{ip\theta_g} \frac{\det(\dot{R}^{L_G}_{Z^0}) \operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(Z^0)}]}{\det\left(1-\exp(-2u\dot{R}^{L_G}_{Z^0})\right)} e^{-u\mathscr{L}^{\perp}_{Z^0}}(Z^{\perp}, Z^{\perp}) \otimes g_E(Z^0). \end{aligned} \tag{2.6.33}$$

Using (2.6.32) and (2.6.33), we get as in (2.6.13)-(2.6.21):

$$\begin{aligned}\limsup_{p\to+\infty} p^{-n+d} \int_U \psi(v) \operatorname{Tr}_q \Big[(P_G e^{-\frac{u}{p}D_p^2} P_G)(v,v) \Big] dv_M(v) \\ \leq \frac{1}{(2\pi)^{n-d}} \frac{1}{|G^0|} \sum_{g\in G^0} \prod_{i=1}^d \sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}} \times \\ \int_{M_G} \psi(x) \frac{\det(\dot{R}^{L_G}_x) \sum_{j=0}^q (-1)^{q-j} \operatorname{Tr}_{\Lambda^{0,j}}[e^{2u\omega_{G,d}(x)}]}{\det\left(1-\exp(-2u\dot{R}^{L_G}_x)\right)} e^{ip\theta_g} \operatorname{Tr}^E[g_E(x)] dv_{M_G}(x) \\ \leq (-1)^{q-r} \int_{M_G(\leq q-r)} \psi(x) \det\Big(\frac{\dot{R}^{L_G}_x}{2\pi}\Big) \frac{1}{|G^0|} \Big(\sum_{g\in G^0} e^{ip\theta_g} \operatorname{Tr}^E[g_E(x)] \Big) dv_{M_G}(x) \\ = (-1)^{q-r} \dim(L^p\otimes E)^{G^0} \int_{M_G(\leq q-r)} \psi(x) \det\Big(\frac{\dot{R}^{L_G}_x}{2\pi}\Big) dv_{M_G}(x). \end{aligned} \tag{2.6.34}$$

Finally, we take some functions ψ_k as ψ above and such that $\sum_k \psi_k = 1$ in a neighborhood of M_G in B and we apply (2.6.34) for those ψ_k. We get Theorem 2.1.5 by taking the sum over k of the obtained estimates and using Theorem 2.1.7 and (2.6.9).

2.6.3 The other isotypic components of the cohomology

In this subsection, we show how to use Theorem 2.1.5 to get estimates on the other isotypic components of the cohomology $H^\bullet(M, L^p \otimes E)$.

Let $\mathcal{V}_\gamma$ be the finite dimensional irreducible representation of G with highest weight γ.

For a representation F of G, we denote by F_γ its isotopic component associated with γ. Then we have

$$\begin{aligned} H^\bullet(M, L^p\otimes E)_\gamma &= \mathcal{V}_\gamma \otimes \operatorname{Hom}_G(\mathcal{V}_\gamma, H^\bullet(M, L^p\otimes E)) \\ &= \mathcal{V}_\gamma \otimes \big(H^\bullet(M, L^p\otimes E)\otimes \mathcal{V}_\gamma^*\big)^G \\ &= \mathcal{V}_\gamma \otimes H^\bullet(M, L^p\otimes E\otimes \mathcal{V}_\gamma^*)^G, \end{aligned} \tag{2.6.35}$$

where $\mathcal{V}_\gamma^*$ is viewed as a trivial bundle over M.

By Theorem 2.1.5 applied replacing E by $E\otimes\mathcal{V}_\gamma^*$ and (2.6.35) we have as $p\to+\infty$,

$$\begin{aligned}\sum_{j=0}^q (-1)^{q-j} \dim H^j(M, L^p\otimes E)_\gamma \\ \leq \dim \mathcal{V}_\gamma \dim(L^p\otimes E\otimes\mathcal{V}_\gamma^*)^{G^0} \frac{p^{n-d}}{(n-d)!} \int_{M_G(\leq q-r)} (-1)^{q-r}\omega_G^{n-d} + o(p^{n-d}), \end{aligned} \tag{2.6.36}$$

with equality for $q = n$.

In particular, we get the weak inequalities

$$\begin{aligned} &\dim H^q(M, L^p \otimes E)_\gamma \\ &\qquad \leq \dim \mathcal{V}_\gamma \dim(L^p \otimes E \otimes \mathcal{V}_\gamma^*)^{G^0} \frac{p^{n-d}}{(n-d)!} \int_{M_G(q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}). \end{aligned} \tag{2.6.37}$$

Chapter 3

The asymptotic of the holomorphic analytic torsion forms

3.1 Introduction

The holomorphic analytic torsion was defined in [54] by Ray and Singer as the complex analogue of its real version for flat vector bundles. It is obtained by regularizing the determinant of the Kodaira Laplacian of holomorphic vector bundles on a compact complex manifold. It appears in the study by Bismut-Gillet-Soulé of the determinant of the fiberwise cohomology of a holomorphic fibration in [12].

Analytic torsion has an extension in the family setting: the analytic torsion forms, defined in various degrees of generality by Bismut-Gillet-Soulé [11], Bismut-Köhler [13] and Bismut [9]. The 0-degree component of these forms is the analytic torsion of Ray-Singer along the fiber. The analytic torsion forms have found many applications, especially because it was introduced, by Gillet and Soulé in particular, as the analytic counterpart of the direct image in Arakelov geometry. In deed, the torsion appear in the arithmetic Riemann-Roch theorem [35] and the torsion forms in the arithmetic Riemann-Roch-Grothendieck theorem in higher degrees [34].

Analytic torsion has also other extensions. In [39], Köhler and Roessler have used an equivariant version of the torsion to prove a Rieamann-Roch theorem for the equivariant arithmetic K_0-theory. Recently in [23], Burgos Gil, Freixas i Montplet and Liţcanu have extended the holomorphic analytic torsion classes to arbitrary projective morphisms between smooth algebraic complex varieties by giving an axiomatic definition. They also classified the theories they obtained. In [22], they use their theory of generalized torsion classes to prove the arithmetic Grothendieck-Riemann-Roch theorem in the case of general projective morphisms between regular arithmetic varieties.

In [17], Bismut and Vasserot computed the asymptotic of the analytic torsion associated with increasing powers of a positive line bundle, using the heat kernel method of [6] (see also [46, Sect. 5.5]). They also extended their result in [18], in the case where the powers of the line bundle are replaced by the symmetric powers of a positive bundle using a trick due to Getzler [33]. These asymptotics have played an important role in a result of arithmetic ampleness by Gillet and Soulé [35] (see also [59, Chp VIII]).

In this chapter, we give the family versions at the level of forms of the results Bismut and Vasserot for the analytic torsion forms.

We first consider the case of torsion forms of a fibration associated with increasing powers of a given positive line bundle which is positive along the fiber. This correspond to [17]. We will use a similar strategy as in that paper, but some additional difficulties appear due to the horizontal differential forms appearing in the Bismut superconnection (compared to the Kodaira Laplacian) used in the definition of the torsion forms. Indeed, the operators we are dealing with here have a nilpotent part (i.e., the part in positive degree along the basis) that must be taken into account, especially when estimating resultants or heat kernels. Moreover, to give the asymptotic formula we have to compute explicitly super-traces of terms involving an exponential coupling horizontal forms and vertical Clifford variables, which makes the computation much more complicated as in [17]. Note also that in all our results of smooth convergence, we have to take into account the derivatives along the basis.

Next, we consider the case of torsion forms of a fibration associated with the direct image of powers of a line bundle on a bigger manifold. We have to make some partial positivity assumption on the line bundle. This generalize [18] in two ways. Firstly we work in the family setting. Secondly it is easy to see that the results of [18] apply in fact to the direct image of powers of a line bundle on a bigger manifold given by a principal G-bundle with G compact and connected. Here we do not assume that this is the case, and as a consequence, we cannot use the same trick as in [18] to reduce the problem to our first result. In the same vein, when the basis is compact Kähler and the fibration is Kähler , we show how to use our first result coupled with [13] and [41] to get simply the asymptotic modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$, but in general this method cannot work.

In the general case, we thus use the same heat kernel approach as in our first result. However here, in addition to the difficulties pointed out above, we have to deal with the fact that the dimension of the bundle we are working with grows to infinity. In particular, we cannot hope to have a limiting operator for the rescaled operator, nor limitings coefficients in the development of the heat kernel, and in all our proofs we have to make uniform estimates on spaces that change. To overcome these issues, we will draw inspiration from [15, 16] and use the formalism of Toeplitz operators of [46]. The idea is to use the operator norm on matrices to have uniform boundedness properties of Toeplitz operators, and to replace the convergence to limiting objects by an approximation by objects with Toeplitz coefficients.

We now give more details about our results. Let M and B be two complex manifolds. Let $\pi\colon M \to B$ be a holomorphic fibration with compact fiber X of dimension n. We denote by TX the holomorphic tangent bundle to the fiber, and $T_{\mathbb{R}}X$ the real tangent bundle. We denote by $T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes \mathbb{C}$ the complexified tangent bundle, and $T^{(1,0)}X, T^{(0,1)}X \subset T_{\mathbb{C}}X$ the $\pm\sqrt{-1}$-eigenspace of the complex structure $J^{T_{\mathbb{R}}X}$ of the fiber. Recall that we have a canonical isomorphism $TX \simeq T^{(1,0)}X$. In the sequel, we will use the same notations for all the other tangent bundles.

Let (π, ω) be a structure of Hermitian fibration in the sense of Section 3.2.1, i.e., ω is a smooth $(1,1)$-form on M which induces a Hermitian metric h^{TX} along the fibers.

Let (ξ, h^{ξ}) be a holomorphic Hermitian vector bundle on M, and let (L, h^L) be a holomorphic Hermitian line bundle on M. We denote the curvature of the Chern connection of L by R^L, and we make the following basic assumption:

Assumption 3.1.1. *The (1,1)-form $\sqrt{-1}R^L$ is positive along the fibers, which means that for any $0 \neq U \in T^{(1,0)}X$, we have*

$$R^L(U, \overline{U}) > 0. \tag{3.1.1}$$

Let $\dot{R}^{X,L} \in \mathrm{End}(TX)$ be the Hermitian matrix such that for any $U, V \in T^{(1,0)}X$,

$$R^L(U, \overline{V}) = \langle \dot{R}^{X,L} U, V \rangle_{h^{TX}}. \tag{3.1.2}$$

By Assumption 3.1.1, $\dot{R}^{X,L}$ is positive definite.

For $p \in \mathbb{N}$, let L^p be the p^{th} tensor power of L. We assume that there is a $p_0 \in \mathbb{N}$ such that the direct image $R^i\pi_*(\xi \otimes L^p)$ is locally free for all $p \geq p_0$ and $i \in \{1, \dots, n\}$, and vanishes for $i > 0$.

In the sequel, all results holds for $p \geq p_0$, and we will not repeat this hypothesis.

Remark 3.1.2. If the basis B is compact, then Assumption 3.1.1 implies that for p large enough the direct image $R^\bullet\pi_*(\xi \otimes L^p)$ is automatically locally free, and that $R^i\pi_*(\xi \otimes L^p) = 0$ for $i > 0$.

We endow $\xi \otimes L^p$ with the metric $h^{\xi \otimes L^p}$ induced by h^ξ and h^L. We can then define (see Section 3.2) the analytic torsion forms $\mathscr{T}(\omega, h^{\xi \otimes L^p})$ associated with (π, ω) and $(\xi \otimes L^p, h^{\xi \otimes L^p})$.

Let

$$\Theta^M = \frac{\sqrt{-1}}{2\pi} R^L \quad \text{and} \quad \Theta^X = \frac{\sqrt{-1}}{2\pi} R^L|_{T_\mathbb{R}X \times T_\mathbb{R}X} \tag{3.1.3}$$

be the first Chern form of (L, h^L) and $(L|_X, h^{L|_X})$ respectively.

If α is a form on B, we denote by $\alpha^{(k)}$ its component of degree k. We can now state our first main result, which is the extension of [17] in the family case:

Theorem 3.1.3. *Let $k \in \{0, \dots, \dim B\}$. Then the component of degree $2k$ of the torsion form $\mathscr{T}(\omega, h^{\xi \otimes L^p})$ associated with ω and $h^{\xi \otimes L^p}$ have the following asymptotic as $p \to +\infty$:*

$$p^{-k}\mathscr{T}(\omega, h^{\xi \otimes L^p})^{(2k)} = \frac{\operatorname{rk}(\xi)}{2} \left(\int_X \log \left[\det \left(\frac{p\dot{R}^{X,L}}{2\pi} \right) \right] e^{\Theta^M + (p-1)\Theta^X} \right)^{(2k)} + o(p^n), \tag{3.1.4}$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B.

We now turn to our second result. Let N, M and B be three complex manifolds. Let $\pi_1 \colon N \to M$ and $\pi_2 \colon M \to B$ be holomorphic fibrations with compact fiber Y and X respectively. Then $\pi_3 := \pi_2 \circ \pi_1 \colon N \to B$ is a holomorphic fibration, whose compact fiber is denoted by Z. We denote by n_X (resp. n_Y, n_Z) the complex dimension of X (resp. Y, Z). Note that $\pi_1|_Z \colon Z \to X$ is a holomorphic fibration with fiber Y. This is summarized in the following diagram:

$$\begin{tikzcd}
Y \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"] & N \arrow[d, "\pi_1"] \arrow[dr, "\pi_3"] & \\
& X \arrow[r] & M \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

We suppose that we are given (π_2, ω^M) a structure of Hermitian fibration (see Section 3.2.1). We denote by $T_B^H M = TX^{\perp, \omega^M}$ the corresponding horizontal space.

Let (ξ, h^ξ) be a holomorphic Hermitian vector bundle on M, and let (η, h^η) be a holomorphic Hermitian vector bundle on N. Let (L, h^L) be a holomorphic Hermitian line bundle on N. We denotes its Chern connection by ∇^L, and the corresponding curvature by R^L.

As above, we make a positivity assumption on L:

Assumption 3.1.4. *The (1,1)-form $\sqrt{-1}R^L$ is positive along the fibers of π_3, that is for any $0 \neq U \in TZ$, we have*

$$R^L(U, \overline{U}) > 0. \tag{3.1.5}$$

In particular, $\frac{\sqrt{-1}}{2\pi}R^L$ enables us to define metrics $g^{T_{\mathbb{R}}Z}$ and $g^{T_{\mathbb{R}}Y}$ on $T_{\mathbb{R}}Z$, and $T_{\mathbb{R}}Y$ (see (3.4.1)).

We assume that there is $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, the direct image $R^\bullet\pi_{1,*}(\eta\otimes L^p)$ is locally free and $R^i\pi_{1,*}(\eta \otimes L^p) = 0$ for $i > 0$. Then for $p \geq p_0$,

$$F_p := H^0(Y, (\eta \otimes L^p)|_Y) \tag{3.1.6}$$

is a $\mathbb{Z}$-graded holomorphic vector bundle on M (see Section 3.2.3), endowed with the L^2 metric h^{F_p} induced by h^η, h^L and $g^{T_{\mathbb{R}}Y}$.

For $p \geq p_0$, we also assume that the direct images $R^\bullet\pi_{2,*}(\xi \otimes F_p)$ and $R^\bullet\pi_{3,*}(\pi_1^*\xi \otimes \eta \otimes L^p)$ are locally free. Then for all $i \geq 0$,

$$R^i\pi_{2,*}(\xi \otimes F_p) \simeq R^i\pi_{3,*}(\pi_1^*\xi \otimes \eta \otimes L^p). \tag{3.1.7}$$

Remark 3.1.5. If the basis B is compact, then Assumption 3.1.4 implies the existence of p_0 such that for $p \geq p_0$ the above conditions are satisfied: the direct images $R^\bullet\pi_{1,*}(\xi \otimes L^p)$, $R^\bullet\pi_{2,*}(\xi \otimes F_p)$ and $R^\bullet\pi_{3,*}(\pi_1^*\xi \otimes \eta \otimes L^p)$ are locally free and moreover they vanish in positive degrees. In particular,

$$H^\bullet(X, (\xi \otimes F_p)|_X) = H^0(X, (\xi \otimes F_p)|_X) \simeq H^0(Z, (\pi_1^*\xi \otimes \eta \otimes L^p)|_Z). \tag{3.1.8}$$

Here again, all results in the sequel holds for $p \geq p_0$, and we will not repeat this hypothesis.

We endow $\xi \otimes F_p$ with the metric $h^{\xi\otimes F_p}$ induced by h^ξ and h^{F_p}. Then we can construct as in Section 3.2 the holomorphic analytic torsion associated with ω^M and $(\xi \otimes F_p, h^{\xi\otimes F_p})$. We denote it by $\mathscr{T}(\omega^M, h^{\xi\otimes F_p})$.

Let

$$T_B^H N = (TZ)^\perp, \quad T_M^H N = (TY)^\perp, \tag{3.1.9}$$

where the orthogonal complements are taken with respect to R^L. Then

$$T_X^H Z := T_M^H N \cap TZ \tag{3.1.10}$$

is the orthogonal complement of TY in TZ. Moreover,

$$T_B^H N \simeq \pi_3^* TB, \quad T_M^H N \simeq \pi_1^* TM \quad \text{and} \quad T_X^H Z \simeq \pi_1^* TX. \tag{3.1.11}$$

Remark 3.1.6. Note that $(\pi_1, g^{T_{\mathbb{R}}Y}, T_M^H N)$ and $(\pi_1|_Z, g^{T_{\mathbb{R}}Y}, T_X^H Z)$ define Kähler fibrations in the sense of Section 3.2.5.

Let $\dot{R}^{X,L} \in \pi_3^*\mathrm{End}(TX)$ be the Hermitian matrix such that for any $U, V \in TX$, if we denote their horizontal lifts by $U^H, V^H \in T_X^H Z$, then

$$R^L(U^H, \overline{V}^H) = \langle \dot{R}^{X,L} U, V\rangle_{h^{TX}}. \tag{3.1.12}$$

By Assumption 3.1.4, $\dot{R}^{X,L}$ is positive definite.

Let

$$\Theta^N = \frac{\sqrt{-1}}{2\pi} R^L \quad \text{and} \quad \Theta^Z = \frac{\sqrt{-1}}{2\pi} R^L|_{T_{\mathbb{R}}Z \times T_{\mathbb{R}}Z}. \tag{3.1.13}$$

We can now state the second main result of this chapter, which is an extension of Theorem 3.1.3, and the family version of [18] (see the introduction of Section 3.4).

Theorem 3.1.7. *Let $k \in \{0, \dots, \dim B\}$. Then the component of degree $2k$ of the torsion form $\mathscr{T}(\omega, h^{\xi\otimes F_p})$ associated with ω^M and $h^{\xi\otimes F_p}$ have the following asymptotic as $p \to +\infty$:*

$$p^{-k}\mathscr{T}(\omega^M, h^{\xi\otimes F_p})^{(2k)} = \frac{\mathrm{rk}(\xi)\mathrm{rk}(\eta)}{2}\left(\int_Z \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right] e^{\Theta^N+(p-1)\Theta^Z}\right)^{(2k)} + o(p^{n_Z}), \tag{3.1.14}$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B.

As explained above and in Section 3.4, we will use the formalism of Toeplitz operator to prove this theorem.

We now recall the definition given in [46, Def. 7.2.1] of a Toeplitz operator.

Let $b \in B$. For $x \in X_b := \pi_1^{-1}(b)$, let $P_{p,x}$ be the orthogonal projection

$$P_{p,x}\colon L^2(Y_x, \eta\otimes L^p) \to H^0(Y_x, \eta\otimes L^p), \tag{3.1.15}$$

where $Y_x := \pi_1^{-1}(x)$.

Definition 3.1.8. A *Toeplitz operator* on Y_x is a family of operators $T_p \in \mathrm{End}(L^2(Y_x, \eta\otimes L^p))$ satisfying the following two properties:

(i) for any $p \in \mathbb{N}$, we have
$$T_p = P_{p,x}T_pP_{p,x}; \tag{3.1.16}$$

(ii) there exists a sequence $f_r \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta))$ such that for any $k \in \mathbb{N}$ there is a constant $C_k > 0$ with
$$\left\| T_p - \sum_{r=0}^{k} p^{-r}P_{p,x}f_rP_{p,x}\right\|_\infty \leq C_k p^{-k-1}. \tag{3.1.17}$$

In the course of the proof of Theorem 3.1.7, we will prove an important result which is that the heat kernel of the Bismut superconnection is asymptotic to a family of Toeplitz operator. Let us give some detail about this result. Let $B_{u,p}$ be the Bismut superconnection associated with ω^M and $(\xi\otimes F_p, h^{\xi\otimes F_p})$ (see Section 3.2.2). Then by Theorem 3.2.8, $B_{u,p}^2$ is a fiberwise elliptic second order differential operator. Let $\exp(-B^2_{p,u/p})$ be the corresponding heat kernel. For $b \in B$, let $\exp(-B^2_{p,u/p})(x,x')$ be the smooth Schwartz kernel of $\exp(-B^2_{p,u/p})$ with respect to $dv_{X_b}(x')$. Then

$$\exp(-B^2_{p,u/p})(x,x) \in \mathrm{End}\left(\Lambda^\bullet_b(T^*_\mathbb{R}B)\otimes\left(\Lambda^{0,\bullet}(T^*X_b)\otimes\xi\otimes F_p\right)\right). \tag{3.1.18}$$

For $a > 0$, ψ_a is the automorphism of $\Lambda(T^*_\mathbb{R}B)$ such that if $\alpha \in \Lambda^q(T^*_\mathbb{R}B)$, then

$$\psi_a\alpha = a^q\alpha. \tag{3.1.19}$$

Let Ω_u be the form defined in (3.4.172). Then we show that

Theorem 3.1.9. *Let $k \in \mathbb{N}$. As $p \to +\infty$, uniformly as u varies in a compact subset of $\mathbb{R}^*_+$, we have the following asymptotic for for the operator norm on* $\mathrm{End}\left(\Lambda^\bullet_b(T^*_\mathbb{R}B)\otimes(\Lambda^{0,\bullet}(T^*X_b)\otimes\xi\otimes F_p)\right)$ *and the operator norm of the derivatives up to order k in the variable $(b,x) \in M$:*

$$\begin{aligned}\psi_{1/\sqrt{p}}&\exp(-B^2_{p,u/p})(x,x)\\ &= \frac{p^{n_X}}{(2\pi)^{n_X}}P_{p,x}e^{-\Omega_{u,(x,\cdot)}}\frac{\det(\dot{R}^{X,L}_{(x,\cdot)})}{\det\left(1-\exp(-u\dot{R}^{X,L}_{(x,\cdot)})\right)}\otimes\mathrm{Id}_{\xi_x}P_{p,x} + o(p^{n_X}).\end{aligned} \tag{3.1.20}$$

Here the dot symbolize the variable in Y_x.

Remark 3.1.10. Note that in the proof of Theorem 3.1.9 which we give in this chapter, we do not use the assumption that L is positive along the fiber Z, but only along the fiber Y.

This chapter is organized as follows. In Section 3.2 we recall the definition given in [9] of the analytic torsion forms, in Section 3.3 we give the asymptotic of the torsion forms associated with increasing powers of a given line bundle and in Section 3.4 we give the asymptotic of the torsion forms associated with the direct image of powers of a line bundle on a bigger manifold. Sections 3.3 and 3.4 begin with introductions where the reader can find the notations, assumptions and statements of the results.

3.2 The holomorphic analytic torsion forms

In this section, following [9, Chap. 3-4], we will define the holomorphic analytic torsion forms associated to a holomorphic Hermitian (non-necessarily Kähler) fibration. This section is organized as follows. In Subsection 3.2.1 we define Hermitian fibrations, In Subsection 3.2.2 we recall the definition of the Bismut superconnection associated with a Hermitian fibration and give the formula for its square, in Subsection 3.2.3, we introduce the cohomology of the fiber as a bundle on the basis and its Chern connection, in Subsection 3.2.4 we define the analytic torsion forms and finally in Subsection 3.2.5 we recall the definition of a Kähler fibration and we specialize the above constructions in this case.

3.2.1 A Hermitian fibration

Let M and B be two complex manifolds of respective dimension m and ℓ. Let $\pi\colon M \to B$ be a holomorphic fibration with n-dimensional compact fiber X. Recall that we denote by TM (resp. TB) the holomorphic tangent bundle of M (resp. B), and by TX the relative holomorphic tangent bundle TM/B. We denote the real tangent bundles by $T_{\mathbb{R}}M$, etc. and their complexification by $T_{\mathbb{C}}M$, etc.

Let $J^{T_{\mathbb{R}}X}$ be the complex structure on $T_{\mathbb{R}}X$, and let ω be a smooth real (1,1)-form on M. Let

$$\omega^X = \omega|_{T_{\mathbb{R}}X\times T_{\mathbb{R}}X}. \tag{3.2.1}$$

We assume that the formula

$$\langle \cdot\,,\cdot \rangle_{g^{T_{\mathbb{R}}X}} := \omega^X(J^{T_{\mathbb{R}}X}\cdot\,,\cdot) \tag{3.2.2}$$

defines a Riemannian structure on $T_{\mathbb{R}}X$. We denote by h^{TX} the associated Hermitian structure on TX.

Let $T^HM \subset TM$ be the orthogonal bundle to TX in TM with respect to ω, and $T^H_{\mathbb{R}}M \subset T_{\mathbb{R}}M$ be the corresponding real vector bundle. Then we have the isomorphism of smooth vector bundles

$$T^HM \simeq \pi^*TB, \quad \text{and} \quad TM = T^HM \oplus TX. \tag{3.2.3}$$

If $U \in T_{\mathbb{R}}B$, we denote by U^H its lift in $T^H_{\mathbb{R}}M$.

The identifications (3.2.3) yields to the isomorphism

$$\Lambda^\bullet(T^*_{\mathbb{R}}M) \simeq \pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B)\widehat{\otimes}\Lambda^\bullet(T^*_{\mathbb{R}}X). \tag{3.2.4}$$

Here, and in this whole chapter, the $\widehat{\otimes}$ denotes the graded tensor product.

Let

$$\omega^H = \omega|_{T^H_{\mathbb{R}}M\times T^H_{\mathbb{R}}M}. \tag{3.2.5}$$

We extend ω^X and ω^H (by 0) to $T^H_{\mathbb{R}}M \oplus T_{\mathbb{R}}X$. Then

$$\omega = \omega^X + \omega^H. \tag{3.2.6}$$

We call the data (π, ω) a *Hermitian fibration.*

3.2.2 The Bismut superconnection of a Hermitian fibration

Let (π, h^{TX}, ω) be a Hermitian fibration.

Let $g^{T_{\mathbb{R}}B}$ be a Riemannian metric on B, and let $g^{T_{\mathbb{R}}M}$ be the metric on M induced by $g^{T_{\mathbb{R}}B}$, $g^{T_{\mathbb{R}}Z}$ and the decomposition (3.2.3). Ultimately, the objects we will define will not depend an the choice of $g^{T_{\mathbb{R}}B}$.

Let (ξ, h^ξ) be a holomorphic Hermitian vector bundle on M. Let ∇^{TX} and ∇^ξ be the Chern connections on (TX, h^{TX}) and (ξ, h^ξ). We denote their curvature by R^{TX} and L^ξ respectively. Let $\nabla^{\Lambda^{0,\bullet}}$ be the connexion induced by ∇^{TX} on $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$, and $\nabla^{\Lambda^{0,\bullet}\otimes\xi}$ be the connexion on $\Lambda^{0,\bullet}(T^*X)\otimes\xi$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^ξ.

Definition 3.2.1. For $0 \le p \le \dim X$, and $b \in B$, set

$$E^k_b = \mathscr{C}^\infty\left(X_b, \left(\Lambda^{0,k}(T^*X)\otimes\xi\right)|_{X_b}\right). \tag{3.2.7}$$

Also set

$$E_b = \bigoplus_{k=0}^{\dim X} E^k_b, \quad E^+_b = \bigoplus_{k \text{ even}} E^k_b, \text{ and } \quad E^-_b = \bigoplus_{k \text{ odd}} E^k_b. \tag{3.2.8}$$

As in [4] or [11], we can think of the E_b's as the fibers of a $\mathbb{Z}$-graded infinite dimensional vector bundle E on B. In this case, smooth sections of E on B are identified with smooth sections of $\Lambda^{0,\bullet}(T^*X)\otimes\xi$ on M.

Let dv_{X_b} be the volume element of $(X_b, h^{TX}|_{X_b})$. Let $\langle\cdot,\cdot\rangle$ be the Hermitian product on E associated to h^{TX} and h^ξ:

$$\langle s, s'\rangle_b = \frac{1}{(2\pi)^{\dim X}}\int_{X_b}\langle s, s'\rangle_{\Lambda^{0,\bullet}\otimes\xi}(x)dv_{X_b}(x). \tag{3.2.9}$$

Definition 3.2.2. For $U \in T_{\mathbb{R}}B$ and s a smooth section of E on B, set

$$\nabla^E_U = \nabla^{\Lambda^{0,\bullet}\otimes\xi}_{U^H}s. \tag{3.2.10}$$

We extend ∇^E to an operator on $\mathscr{C}^\infty\big(M, \pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B)\otimes\Lambda^{0,\bullet}(T^*X)\otimes\xi\big)$, which will be again denoted by ∇^E. Let $\nabla^{E'}$ and $\nabla^{E''}$ be the holomorphic and anti-holomorphic part of the connection ∇^E.

Note that ∇^E does not necessarily preserve the Hermitian product (3.2.9) on E.

For $b \in B$, let $\bar\partial^{X_b}$ be the Dolbeault operator acting on E_b and let $\bar\partial^{X_b,*}$ be its formal adjoint with respect to the Hermitian product (3.2.9). Set

$$D^{X_b} = \bar\partial^{X_b} + \bar\partial^{X_b,*}. \tag{3.2.11}$$

Let $C(T_{\mathbb{R}}X)$ be the Clifford algebra of $(T_{\mathbb{R}}X, g^{T_{\mathbb{R}}X})$. The bundle $\Lambda^{0,\bullet}(T^*X) \otimes \xi$ is a $C(T_{\mathbb{R}}X)$-Clifford module: if $U \in TX \simeq T^{(1,0)}X$, denote by $U^* \in T^{*(0,1)}X$ its dual for the metric h^{TX}, and then

$$c(U) = \sqrt{2}U^* \wedge \text{ and } c(\overline{U}) = -\sqrt{2}i_{\overline{U}}. \tag{3.2.12}$$

Let $P^{T_{\mathbb{R}}X}$ be the projection $T_{\mathbb{R}}M = T_{\mathbb{R}}^H M \oplus T_{\mathbb{R}}X \to T_{\mathbb{R}}X$. For $U, V \in \mathscr{C}^\infty(B, T_{\mathbb{R}}B)$ set

$$T(U,V) = -P^{T_{\mathbb{R}}X}[U^H, V^H]. \tag{3.2.13}$$

Definition 3.2.3. Let $f_1, \dots, f_{2n}$ be a basis of $T_{\mathbb{R}}B$ and $f^1, \dots, f^{2n}$ its dual basis. Set

$$c(T) = \frac{1}{2}\sum_{\alpha,\beta} f^\alpha f^\beta c\left(T(f_\alpha, f_\beta)\right), \tag{3.2.14}$$

which is a section of $\left[\Lambda(T_{\mathbb{R}}^*B)\widehat{\otimes}\mathrm{End}\left(\Lambda^{0,\bullet}(T^*X)\otimes\xi\right)\right]^{\mathrm{odd}}$.

Let $T^{(1,0)}$ and $T^{(0,1)}$ be the component of the $(1,1)$ form T in $T^{(1,0)}X$ and $T^{(0,1)}X$ respectively. We define $c(T^{(1,0)})$ and $c(T^{(0,1)})$ as in (3.2.14), so that

$$c(T) = c(T^{(1,0)}) + c(T^{(0,1)}). \tag{3.2.15}$$

Let γ be the one form on $T_{\mathbb{R}}B$ such that

$$\mathcal{L}_{A^H} dv_X = \gamma(A) dv_X. \tag{3.2.16}$$

We assume temporarily that $\det(TX)$ has a square root λ. Equivalently, $T_{\mathbb{R}}X$ is equipped with a spin structure. Then λ is a holomorphic Hermitian vector bundle on M. Let ∇^λ be the corresponding Chern connection. Let

$$\mathcal{S}^{TX} = \Lambda^{0,\bullet}(T^*X)\widehat{\otimes}\lambda^* \tag{3.2.17}$$

be the associated $(T_{\mathbb{R}}X, g^{T_{\mathbb{R}}X})$-spinor bundle. Let $\nabla^{\mathcal{S}^{TX},LC}$ be the connection on $\mathcal{S}^{TX}$ induced by $\nabla^{T_{\mathbb{R}}X,LC}$, the Levi-Civita connection of $T_{\mathbb{R}}X$. Finally, let $\nabla^{\Lambda^{0,\bullet},LC}$ be the connection on $\Lambda^{0,\bullet}(T^*X)$ induced by $\nabla^{\mathcal{S}^{TX},LC}$ and ∇^λ, and let $\nabla^{\Lambda^{0,\bullet}\otimes\xi,LC}$ be the connection induced by $\nabla^{\Lambda^{0,\bullet},LC}$ and ∇^ξ on $\Lambda^{0,\bullet}(T^*X)\otimes\xi$.

Note that, as $\det(TX)$ has always locally a square root, the connection $\nabla^{\Lambda^{0,\bullet}\otimes\xi,LC}$ is in fact always defined.

The reader should be careful about the fact that in [9], the Clifford algebra $C(T_{\mathbb{R}}X)$ is constructed with respect to $g^{T_{\mathbb{R}}X}/2$, so that our formulas will differ from those of [9] by some powers of $1/\sqrt{2}$.

Let $(e_1, \dots, e_{2n})$ be an orthonormal frame of $T_{\mathbb{R}}X$.

Definition 3.2.4. We follow here [9, Defs. 3.7.2, 3.7.4 and 3.7.5].

1. Let
$$D^{X,LC} = c(u_i)\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes\xi,LC} \tag{3.2.18}$$
be the Dirac operator of the fiber.

2. For $U \in T_{\mathbb{R}}B$ and s a smooth section of E, let
$$\nabla_U^{E,LC} s = \nabla_{U^H}^{\Lambda^{0,\bullet}\otimes\xi,LC} s + \frac{1}{2}\gamma(U)s. \tag{3.2.19}$$

3. Finally, let
$$A^{LC} = \nabla^{E,LC} + \frac{1}{\sqrt{2}} D^{X,LC} - \frac{c(T)}{2\sqrt{2}}. \tag{3.2.20}$$
This superconnection on E is called the *Bismut superconnection.*

Let $(e^1, \dots, e^{2n})$ be the dual frame of $(e_1, \dots, e_{2n})$. We define a map $\alpha \mapsto \alpha^c$ from $\Lambda(T^*_{\mathbb{R}}X)$ to $C(T_{\mathbb{R}}X)$ by setting for $1 \le i_1 < \dots < i_k \le 2n$:
$$\left(e^{i_1} \wedge \dots \wedge e^{i_k}\right)^c = 2^{-k/2} c(e_{i_1}) \dots c(e_{i_k}). \tag{3.2.21}$$
We extend this map to a map (denoted in the same way) from $\Lambda^\bullet(T^*_{\mathbb{R}}M) \simeq \pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B)\widehat{\otimes}\Lambda^\bullet(T^*_{\mathbb{R}}X)$ to $\pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B)\widehat{\otimes}C(T_{\mathbb{R}}X)$.

Proposition 3.2.5. *The following formula holds*
$$D^X = \frac{1}{\sqrt{2}} D^{X,LC} + \frac{1}{2}\Big(\left(\bar\partial^X - \partial^X\right) i\omega^X\Big)^c. \tag{3.2.22}$$

Proof. See [9, Thm. 3.7.3] or [46, Thm. 1.4.5]. □

Recall that for $a > 0$, ψ_a is the automorphism of $\Lambda(T^*_{\mathbb{R}}B)$ such that if $\alpha \in \Lambda^q(T^*_{\mathbb{R}}B)$, then
$$\psi_a \alpha = a^q \alpha. \tag{3.2.23}$$
By (3.2.3), we may also see ψ_a as an automorphism of $\Lambda(T^{H,*}_{\mathbb{R}}M)$.

We can now define the superconnection of main interest for us.

Definition 3.2.6. For $u > 0$, the *Bismut superconnection* B on E, and its rescaled version B_u are defined by
$$\begin{aligned} B &= A^{LC} + \frac{1}{2}\Big(\left(\bar\partial^M - \partial^M\right) i\omega\Big)^c, \\ B_u &= \sqrt{u}\psi_{1/\sqrt{u}} B \psi_{\sqrt{u}}. \end{aligned} \tag{3.2.24}$$

Then B_u acts on
$$\Omega^\bullet(B, E) := \mathscr{C}^\infty\Big(M, \pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B)\widehat{\otimes}\Lambda^{0,\bullet}(T^*X) \otimes \xi\Big). \tag{3.2.25}$$

Moreover, by [9, (3.3.3), (3.5.17), (3.6.4) and (3.8.1)], the part of degree 0 in $\Lambda^\bullet T_{\mathbb{R}}B$ of B is
$$B^{(0)} = D^X. \tag{3.2.26}$$

Remark 3.2.7. This definition of the elliptic superconnection is not the more natural and correspond in fact to [9, Thm. 3.8.1]. However, for the sake of concision we prefer to define B this way. We refer the reader to [9, Chap. 3] for an other definition of B.

Let $h^{T_{\mathbb{R}}B}$ be a metric on B and let $\nabla^{T_{\mathbb{R}}B,LC}$ be the corresponding Levi-Civita connection. Then $\nabla^{T_{\mathbb{R}}B,LC}$ lifts to a connection $\nabla^{T^H_{\mathbb{R}}M,LC}$ on $T^H_{\mathbb{R}}M$, and we define $\nabla^{T_{\mathbb{R}}M}_s = \nabla^{T^H_{\mathbb{R}}M,LC} \oplus \nabla^{T_{\mathbb{R}}X,LC}$. Let $\nabla^{T_{\mathbb{R}}M,LC}$ be the Levi-Civita connection of M. Set
$$\mathrm{S} = \nabla^{T_{\mathbb{R}}M,LC} - \nabla^{T_{\mathbb{R}}M,LC}_s. \tag{3.2.27}$$

Then S is a one form on M taking values in antisymmetric elements of $\mathrm{End}(T_{\mathbb{R}}M)$. Moreover, by [4, Thm. 1.9], the $(3,0)$-tensor
$$S(\cdot\,,\cdot\,,\cdot\,) = \langle \mathrm{S}(\cdot)\cdot, \cdot\rangle_{h^{T_{\mathbb{R}}M}} \tag{3.2.28}$$

does not depend on $h^{T_{\mathbb{R}}B}$.

Recall that $P^{T_{\mathbb{R}}X}$ is the projector on $T_{\mathbb{R}}X$ defined by the decomposition (3.2.3). Let T be the torsion of $\nabla_s^{T_{\mathbb{R}}M}$. Then by [8, Thm. 1.1], for $U, V \in T_{\mathbb{R}}X$ and $E, F \in T_{\mathbb{R}}B$,

$$\begin{cases} T(U,V) = 0, \\ T(E^H, F^H) = -P^{T_{\mathbb{R}}X}[E^H, F^H], \\ T(E^H, U) = \dfrac{1}{2}(g^{T_{\mathbb{R}}X})^{-1}(\mathcal{L}_{E^H} g^{T_{\mathbb{R}}X})U. \end{cases} \tag{3.2.29}$$

Equations (3.2.13) and (3.2.29) justify that we keep the notation T. We then have an explicit formula for S: for any $U \in T_{\mathbb{R}}X$ and any $V, W \in T_{\mathbb{R}}M$,

$$2S(U,V,W) = \langle T(V,W), P^{T_{\mathbb{R}}X}U\rangle - \langle T(U,V), P^{T_{\mathbb{R}}X}W\rangle - \langle T(W,U), P^{T_{\mathbb{R}}X}V\rangle. \tag{3.2.30}$$

From now on, we will always use latin indices $i, j, \dots$ for the vertical variables, and greek indices $\alpha, \beta, \dots$ for the horizontals variables. Let $\{e_i\}$ be an orthonormal basis of $T_{\mathbb{R}}X$ with dual basis $\{e^i\}$ and $\{f_\alpha\}$ a basis of $T_{\mathbb{R}}B$ with dual basis $\{f^\alpha\}$ (which will be identified with basis of $T_{\mathbb{R}}^H M$ and $(T_{\mathbb{R}}^H M)^*$). For any $(k,0)$-tensor A, we will denote by $A_{a_1,\dots,a_k} = A(e_{a_1}, \dots, e_{a_k})$ where $e_{a_i} = e_j$ or f_α. For instance, if $A \in \Lambda^2(T_{\mathbb{R}}^* M)$,

$$A = \frac{1}{2}A_{i,j}e^i e^j + A_{i,\alpha}e^i f^\alpha + \frac{1}{2}A_{\alpha,\beta}f^\alpha f^\beta. \tag{3.2.31}$$

Let K^X be the scalar curvature of (X, TX). Set

$$L'^\xi = L^\xi + \frac{1}{2}\operatorname{Tr}(R^{TX}). \tag{3.2.32}$$

For $u > 0$, define

$$\nabla_{u,e_i} = \nabla_{e_i}^{\Lambda^{0,\bullet}\otimes\xi, LC} + \frac{1}{\sqrt{2u}}S_{i,j,\alpha}c(e_j)f^\alpha + \frac{1}{2u}S_{i,\alpha,\beta}f^\alpha f^\beta + \frac{1}{2}\psi_{1/\sqrt{u}}\left(i_{e_i}\left(\bar\partial^M - \partial^M\right)i\omega\right)^c\psi_{\sqrt{u}}, \tag{3.2.33}$$

which is a fiberwise connection on $\pi^*\Lambda^\bullet(T_{\mathbb{R}}^*B) \otimes \Lambda^{0,\bullet}(T^*X) \otimes \xi$.

The following theorem is the fundamental Lichnerowicz formula proved in [9, Thm. 3.9.3].

Theorem 3.2.8. *For $u > 0$,*

$$\begin{aligned} B_u^2 = -\frac{u}{2}(\nabla_{u,e_i})^2 + \frac{uK^X}{8} + \frac{u}{4}c(e_i)c(e_j)L'^\xi_{i,j} + \sqrt{\frac{u}{2}}c(e_i)f^\alpha L'^\xi_{i,\alpha} + \frac{f^\alpha f^\beta}{2}L'^\xi_{\alpha,\beta} \\ - u\psi_{1/\sqrt{u}}\left(\bar\partial^M\partial^M i\omega\right)^c\psi_{\sqrt{u}} - \frac{u}{16}\left\| \left(\bar\partial^X - \partial^X\right)i\omega^X\right\|^2_{\Lambda^\bullet(T_{\mathbb{R}}^*X)}. \end{aligned} \tag{3.2.34}$$

Thus, B_u^2 is a fiberwise elliptic second order differential operator. In particular, its heat kernel $\exp(-B_u^2)$ *exists.*

Remark 3.2.9. In this theorem, as in the whole article, we use the usual following notation: if C is a smooth section of $T_{\mathbb{R}}^*X \otimes \operatorname{End}(\Lambda^{0,\bullet}(T^*X)\otimes\xi)$, then

$$\left(\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes\xi} + C(e_i)\right)^2 = \sum_i\left(\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes\xi} + C(e_i)\right)^2 - \nabla^{\Lambda^{0,\bullet}\otimes\xi}_{\sum_i \nabla_{e_i}^{TX}e_i} - C\left(\sum_i \nabla_{e_i}^{TX}e_i\right). \tag{3.2.35}$$

3.2.3 The cohomology of the fiber

We assume that the direct image $R^\bullet\pi_*\xi$ of ξ by π is locally free. For $b \in B$, let $H^\bullet(X_b, \xi|_{X_b})$ be the cohomology of the sheaf of holomorphic sections of ξ over X_b. Then the $H^\bullet(X_b, \xi|_{X_b})$'s form a $\mathbb{Z}$-graded holomorphic vector bundle $H(X, \xi|_X)$ on B and $R^\bullet\pi_*\xi = H^\bullet(X, \xi|_X)$.

For $b \in B$, let $K(X_b, \xi|_{X_b}) = \ker(D^{X_b})$. By Hodge theory, we know that for every $b \in B$

$$H^\bullet(X_b, \xi|_{X_b}) \simeq K^\bullet(X_b, \xi|_{X_b}), \tag{3.2.36}$$

The Hermitian product (3.2.9) on E_b restricts to the right and side of (3.2.36), so h^{TX} and h^ξ induce a metric $h^{H(X,\xi|_X)}$ on the holomorphic vector bundle $H(X, \xi|_X)$, for which the $H^k(X, \xi|_X)$'s are mutually orthogonal.

Definition 3.2.10. Let $\nabla^{H(X,\xi|_X)}$ be the Chern connection on $\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big)$.

For $\overline{U} \in T^{0,1}B$ and $s \in \mathscr{C}^\infty(B, E)$, set $\nabla^{E,u''}_{\overline{U}} = \mathcal{L}_{\overline{U}^H}$. Let $\nabla^{E,u'}$ be the adjoint of $\nabla^{E,u''}$ defined by

$$\langle \nabla^{E,u'} s, s' \rangle = \langle s, \nabla^{E,u''} s' \rangle, \tag{3.2.37}$$

and let $\nabla^{E,u} = \nabla^{E,u'} + \nabla^{E,u''}$ (see [9, Chp. 3]).

Let P^{K_b} be the orthogonal projection form E_b on $K(X_b, \xi|_{X_b})$. We define the connection $\nabla^{K(X,\xi|_X)}$ on $K(X, \xi|_X)$ by

$$\nabla^{K(X,\xi|_X)} = P^K \nabla^{E,u} P^K. \tag{3.2.38}$$

The following proposition is proved in [9, Prop. 4.10.3].

Proposition 3.2.11. *Under the identification* (3.2.36)*, the connections* $\nabla^{H(X,\xi|_X)}$ *and* $\nabla^{K(X,\xi|_X)}$ *agree.*

3.2.4 The analytic torsion forms

Definition 3.2.12. For any complex manifold Z, we denote by Q^Z the vector space of real forms on Z which are sum of forms of type (p, p). We also denote by $Q^{Z,0}$ the subspace of the $\alpha \in Q^Z$ that can be written $\alpha = \partial\beta + \bar{\partial}\gamma$ for some β, γ smooth form on Z.

Let N_V be the number operator defining the $\mathbb{Z}$-grading on $\Lambda^{0,\bullet}(T^*X) \otimes \xi$ and on E.

Definition 3.2.13. For $u > 0$, set

$$N_u = N_V + i\frac{\omega^H}{u}. \tag{3.2.39}$$

Let Φ be the endomorphism of $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ defined by

$$\alpha \in \Lambda^{2k}(T^*_{\mathbb{R}}B) \mapsto (2i\pi)^{-k}\alpha. \tag{3.2.40}$$

Let τ be the involution defining the $\mathbb{Z}_2$-graduation on E. If $H \in \mathrm{End}(E)$ is trace class, we define its *supertrace* $\mathrm{Tr}_s[H]$ by

$$\mathrm{Tr}_s[H] = \mathrm{Tr}[\tau H]. \tag{3.2.41}$$

We extend the supertrace to get an application $\mathrm{Tr}_s \colon \Omega^\bullet(B, E) \to \Omega^\bullet(B)$.

Theorem 3.2.14 (see [9, Thm. 4.5.2]). *For any $u > 0$, the forms $\Phi \operatorname{Tr}_s\left[\exp(-B_u^2)\right]$ and $\Phi \operatorname{Tr}_s\left[N_u \exp(-B_u^2)\right]$ lie in Q^B. Moreover the following identity holds in Q^B*

$$\frac{\partial}{\partial u}\Phi \operatorname{Tr}_s\left[\exp(-B_u^2)\right] = -\frac{1}{u}\frac{\bar\partial\partial}{2i\pi}\Phi \operatorname{Tr}_s\left[N_u \exp(-B_u^2)\right]. \tag{3.2.42}$$

Let $(\alpha_u)_{u\in\mathbb{R}^+}$ and α be smooth forms on B. We say that as $u \to +\infty$ (resp. $u \to 0$), $\alpha_u = \alpha + O(f(u))$, if and only if for any compact set K in B and any $k \in \mathbb{N}$ their exists $C > 0$ such that for every $u \geq 1$ (resp. $u \leq 1$) the norm of all the derivatives of order $\leq k$ of $\alpha_u - \alpha$ over K is bounded by $Cf(u)$.

Theorem 3.2.15 (see [9, Thm. 4.10.4]). *As $u \to +\infty$,*

$$\begin{aligned}\Phi \operatorname{Tr}_s\left[\exp(-B_u^2)\right] &= \Phi \operatorname{Tr}_s\left[\exp(-(\nabla^{H(X,\xi|_X)})^2)\right] + O\left(\frac{1}{\sqrt{u}}\right),\\ \Phi \operatorname{Tr}_s\left[N_u\exp(-B_u^2)\right] &= \Phi \operatorname{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi|_X)})^2)\right] + O\left(\frac{1}{\sqrt{u}}\right).\end{aligned} \tag{3.2.43}$$

Theorem 3.2.16 (see [9, Prop. 4.6.1]). *There exist locally computable forms $(c_j \in Q^B)_{j\geq -m}$ and $(C_j \in Q^B)_{j\geq -m}$ such that for $u \to 0$ and for any $k \in \mathbb{N}$,*

$$\Phi \operatorname{Tr}_s\left[\exp(-B_u^2)\right] = \sum_{j=-m}^{k} c_j u^j + O(u^{k+1}), \tag{3.2.44}$$

and

$$\Phi \operatorname{Tr}_s\left[N_u\exp(-B_u^2)\right] = \sum_{j=-m}^{k} C_j u^j + O(u^{k+1}). \tag{3.2.45}$$

Following [11, Def. 2.19], [13, Def. 3.8] and [9, (4.11.3)], we can now define the analytic torsion forms.

For $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$, by Theorem 3.2.16, we can set

$$\zeta^1(s) = -\frac{1}{\Gamma(s)}\int_0^1 u^{s-1}\Phi\left\{\operatorname{Tr}_s\left[N_u\exp(-B_u^2)\right] - \operatorname{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi|_X)})^2)\right]\right\} du, \tag{3.2.46}$$

and ζ^1 has a meromorphic extension to $\mathbb{C}$, which is holomorphic on $\{|\operatorname{Re}(s)| < 1/2\}$.

Similarly for $s \in \mathbb{C}$, $\operatorname{Re}(s) < 1/2$, Theorem 3.2.15 allows us to define

$$\zeta^2(s) = -\frac{1}{\Gamma(s)}\int_1^{+\infty} u^{s-1}\Phi\left\{\operatorname{Tr}_s\left[N_u\exp(-B_u^2)\right] - \operatorname{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi|_X)})^2)\right]\right\} du. \tag{3.2.47}$$

Here again, ζ^2 has a holomorphic extension on $\{|\operatorname{Re}(s)| < 1/2\}$.

Now, for $s \in \mathbb{C}$, $|\operatorname{Re}(s)| < 1/2$, define the holomorphic function

$$\zeta(s) = \zeta^1(s) + \zeta^2(s). \tag{3.2.48}$$

Definition 3.2.17. The *holomorphic analytic torsion form* is the form

$$\mathscr{T}(\omega, h^\xi) := \frac{\partial}{\partial s}\zeta(0). \tag{3.2.49}$$

The component in the different degrees of $\mathscr{T}(\omega, h^\xi)$ are referred to as the *holomorphic analytic torsion forms.*

Using (3.2.45), we can write

$$\begin{aligned}\mathscr{T}(\omega,h^{\xi}) = &-\int_0^1 \left\{\Phi\,\mathrm{Tr}_s\left[N_u\exp(-B_u^2)\right] - \sum_{j=-m}^{0} C_j u^j\right\}\frac{du}{u}\\ &-\int_1^{+\infty}\Phi\left\{\mathrm{Tr}_s\left[N_u\exp(-B_u^2)\right] - \mathrm{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi|_X)})^2)\right]\right\}\frac{du}{u} \qquad (3.2.50)\\ &+\sum_{j=-m}^{-1}\frac{C_j}{j} + \Gamma'(1)\left(C_0 - \Phi\,\mathrm{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi|_X)})^2)\right]\right).\end{aligned}$$

The following analogue to [13, Thm. 3.9] is proved in [9, Thm. 4.11.2]

Theorem 3.2.18. *The smooth form $\mathscr{T}(\omega,h^{\xi})$ lies in Q^B. Moreover*

$$\frac{\bar{\partial}\partial}{2i\pi}\mathscr{T}(\omega,h^{\xi}) = \mathrm{ch}\left(H(X,\xi|_X),h^{H(X,\xi|_X)}\right) - c_0. \qquad (3.2.51)$$

3.2.5 The case of a Kähler fibration

Following [11, Def. 1.4 and Thm. 1.5], we say that the Hermitian fibration (π,ω) is a Kähler fibration if ω is closed.

We assume in this subsection that ω is closed. Then by [9, Thms. 3.7.1, 3.7.3 and 3.8.1] the superconnection B_u agrees we the one define in [13, Def. 1.7], which is the usual Bismut superconnection.

Therefor, (3.2.24), (3.2.33) and (3.2.34) turn repectively to

$$\begin{cases} B_u = \nabla^E + \sqrt{u}D^X - \dfrac{c(T)}{2\sqrt{2u}},\\ \nabla_{u,e_i} = \nabla_{e_i}^{\Lambda^{0,\bullet}\otimes\xi} + \dfrac{1}{\sqrt{2u}}S_{i,j,\alpha}c(e_j)f^{\alpha} + \dfrac{1}{2u}S_{i,\alpha,\beta}f^{\alpha}f^{\beta} \qquad \text{and}\\ B_u^2 = -\dfrac{u}{2}(\nabla_{u,e_i})^2 + \dfrac{uK^X}{8} + \dfrac{u}{4}c(e_i)c(e_j)L'^{\xi}_{i,j} + \sqrt{\dfrac{u}{2}}c(e_i)f^{\alpha}L'^{\xi}_{i,\alpha} + \dfrac{f^{\alpha}f^{\beta}}{2}L'^{\xi}_{\alpha,\beta}. \end{cases} \qquad (3.2.52)$$

Moreover, [11, Thm. 2.2] sharpens (3.2.44): the forms c_j, for $j \le 0$, can be explicitly computed. For any Hermitian vector bundle (F,h^F) with Chern connection ∇^F and curvature R^F on M, set

$$\begin{aligned}\mathrm{ch}(F,h^F) &= \mathrm{Tr}\left[\exp\left(-\frac{R^F}{2\sqrt{-1}\pi}\right)\right],\\ \mathrm{Td}(F,h^F) &= \det\left(\frac{R^F/2\sqrt{-1}\pi}{\exp(R^F/2\sqrt{-1}\pi)-1}\right).\end{aligned} \qquad (3.2.53)$$

Then by [11, Thm. 2.2] we get

$$\begin{cases} c_j = 0 \text{ for } j<0 \qquad \text{and}\\ c_0 = \displaystyle\int_X \mathrm{Td}(TX,h^{TX})\mathrm{ch}(\xi,h^{\xi}). \end{cases} \qquad (3.2.54)$$

Finally, by [13, Thm. 1.5] ∇^E preserve the metric on E and by [13, Thm. 3.2] we have

$$\nabla^{H(X,\xi|_X)} = P^K\nabla^E P^K. \qquad (3.2.55)$$

Then Theorem 3.2.18 become [13, Thm. 3.9], that is

$$\frac{\bar{\partial}\partial}{2i\pi}\mathscr{T}(\omega, h^{\xi}) = \mathrm{ch}\left(H(X, \xi|_X), h^{H(X,\xi|_X)}\right) - \int_X \mathrm{Td}(TX, h^{TX})\mathrm{ch}(\xi, h^{\xi}). \tag{3.2.56}$$

3.3 The asymptotic of the torsion associated to high power of a line bundle

The purpose of this section is to prove Theorem 3.1.3.

We recall some notations. Let M and B be two complex manifolds. Let $\pi\colon M \to B$ be a holomorphic fibration with compact fiber X of dimension n. We suppose that we are given (π, ω) a structure of Hermitian fibration.

Let (ξ, h^{ξ}) be a holomorphic Hermitian vector bundle on M, and let (L, h^L) be a holomorphic Hermitian line bundle on M. We denote the curvature of the Chern connection of L by R^L. Recall that by Assumption 3.1.1, R^L is assumed to be positive along the fibers. We define

$$\Theta^M = \frac{\sqrt{-1}}{2\pi}R^L \quad \text{and} \quad \Theta^X = \frac{\sqrt{-1}}{2\pi}R^L|_{T_{\mathbb{R}}X\times T_{\mathbb{R}}X}. \tag{3.3.1}$$

We have also assumed that the direct image $R^i\pi_*(\xi \otimes L^p)$ is locally free (for p large). We will use all the constructions of Section 3.2 associated with $(\xi \otimes L^p, h^{\xi\otimes L^p})$ instead of (ξ, h^{ξ}) (where of course $h^{\xi\otimes L^p}$ is induced by h^{ξ} and h^L). The corresponding objects will be denoted by

$$\begin{aligned}
&E^k_{p,b} = \mathscr{C}^{\infty}\left(X_b, \left(\Lambda^{0,k}(T^*X)\otimes\xi\otimes L^p\right)|_{X_b}\right),\\
&\nabla^p = \nabla^{E_p, LC},\\
&\bar{\partial}^p = \text{Dolbeault operator of } E_p,\\
&D_p = \bar{\partial}^p + \bar{\partial}^{p,*},\\
&B_p = \text{associated superconnection as in (3.2.24)},\\
&B_{p,u} = \sqrt{u}\psi_{1/\sqrt{u}}B_p\psi_{\sqrt{u}}.
\end{aligned} \tag{3.3.2}$$

We also denote by $\mathscr{T}(\omega, h^{\xi\otimes L^p})$ the associated analytic torsion forms.

Theorem 3.1.3 is the family version of [17]. The strategy of proof is similar, but differences appear in the proof of the intermediate results due to the horizontal differential forms appearing in B_p^2. One of the first consequence is that, unlike D_p^2, the operator B_p^2 is not self-adjoint, and one has to take a nilpotent part (the part in positive degree along the basis) into account when estimating resolvants or heat kernels (compare for instance the proofs of [17, (20)] and of Theorem 3.3.25). An other consequence is the limit of the heat kernel involves exponential of terms coupling horizontal forms and vertical Clifford variables, which make the computations of the super-traces much more complicated (see Theorem 3.3.26). Note also that in all our results of smooth convergence, we have to take into account the derivatives along the basis B.

To simplify the statements in the following, we will assume that B is compact. However, the reader should be aware of the fact that the constants appearing in the sequel depends on the compact subset of B we are working on.

This section is organized as follows. In Subsection 3.3.1, we show that our problem is local. In Subsection 3.3.2, we rescale the Bismut superconnection and compute the limit operator. In Subsection 3.3.3, we obtain the corresponding convergence of the heat kernel. In Subsection 3.3.4, we prove our main theorem, using two result which are proved in Subsections 3.3.5 and 3.3.6.

3.3.1 Localization of the problem

Fix $b_0 \in B$. In this section, we will work along the fiber X_{b_0}, which will be denoted simply by X.

For $\varepsilon > 0$ and $x_0 \in X$, we denote by $B^X(x_0, \varepsilon)$ and $B^{T_{\mathbb{R},x_0}X}(0, \varepsilon)$ the open balls in X and $T_{\mathbb{R},x_0}X$ with center x_0 and 0 and radius ε respectively. If $\exp^X_{x_0}$ is the exponential map of X, then for ε small enough, $Z \in B^{T_{\mathbb{R},x_0}X}(0, \varepsilon) \mapsto \exp^X_{x_0}(Z) \in B^X(x_0, \varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{\mathbb{R},x_0}X$ with $\mathbb{R}^{2n}$ via an orthonormal basis $\{e_i\}$ of $T_{\mathbb{R},x_0}X$:

$$(Z_1, \ldots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_i Z_i e_i \in T_{\mathbb{R},x_0}X. \tag{3.3.3}$$

From now on, we will always identify $B^{T_{\mathbb{R},x_0}X}(0, \varepsilon)$ and $B^X(x_0, \varepsilon)$.

Let inj^X be the injectivity radius of X and let $\varepsilon \in]0, \mathrm{inj}^X/4[$. Such an ε can be chosen uniformly for b_0 varying in a compact subset of B.

Let $x_1, \ldots x_N$ be points of X such that $\{U_{x_k} = B^X(x_k, \varepsilon)\}_{k=1}^N$ is an open covering of X. On each U_{x_k} we identify ξ_Z, L_Z and $\Lambda^{0,\bullet}(T^*_Z X)$ to ξ_{x_k}, L_{x_k} and $\Lambda^{0,\bullet}(T^*_{x_k}X)$ by parallel transport with respect to ∇^ξ, ∇^L and $\nabla^{\Lambda^{0,\bullet},LC}$ along the geodesic ray $t \in [0,1] \mapsto tZ$. We fixe for each $k = 1, \ldots, N$ an orthonormal basis $\{e_i\}_i$ of $T_{\mathbb{R},x_k}X$ (without mentioning the dependence on k).

We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_k}X$.

We define the vector bundle $\mathbb{E}_p$ over X by

$$\mathbb{E}_p := \Lambda^\bullet_{b_0}(T^*_\mathbb{R} B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi \otimes L^p\right). \tag{3.3.4}$$

Note here that $\Lambda^\bullet_{b_0}(T^*_\mathbb{R} B)$ is a trivial bundle over X.

Let $\{\varphi_k\}_k$ be a partition of unity subordinate to $\{U_{x_k}\}_k$. For $\ell \in \mathbb{N}$, we define a Sobolev norm $||\cdot||_{\boldsymbol{H}^\ell(p)}$ on the ℓ-th Sobolev space $\boldsymbol{H}^\ell(X, \mathbb{E}_p)$ by

$$||s||^2_{\boldsymbol{H}^\ell(p)} = \sum_k \sum_{d=0}^{\ell} \sum_{i_1,\ldots,i_d=1}^{d} ||\nabla_{e_{i_1}} \ldots \nabla_{e_{i_d}}(\varphi_k s)||^2_{L^2}. \tag{3.3.5}$$

Lemma 3.3.1. *For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that for any $p \in N$, $u > 0$ and $s \in \boldsymbol{H}^{2m+2}(X, \mathbb{E}_p)$,*

$$||s||^2_{\boldsymbol{H}^{2m+2}(p)} \leq C_m p^{4m+4} \sum_{j=0}^{m+1} p^{-4j} ||B_p^{2j} s||_{L^2}. \tag{3.3.6}$$

Proof. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to $\nabla^{T_\mathbb{R}X,LC}$ along the curve $t \in [0,1] \mapsto tZ$. Then $\{\widetilde{e}_i\}_i$ is an orthonormal frame of $T_\mathbb{R}X$.

Let Γ^ξ, Γ^L and $\Gamma^{\Lambda^{0,\bullet},LC}$ be the corresponding connection form of ∇^ξ, ∇^L and $\nabla^{\Lambda^{0,\bullet},LC}$ with respect to any fixed frame for ξ, L and $\Lambda^{0,\bullet}(T^*X)$ which is parallel along the curve $t \in [0,1] \mapsto tZ$ under the trivialization on U_{x_k}. Let $\nabla^p_1 = \nabla_1 \otimes 1 + 1 \otimes \nabla^{L^p}$ be the connection on $\Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes \xi$ corresponding to ∇_u in (3.2.33) (with $u = 1$), replacing ξ by $\xi \otimes L^p$. Then on U_{x_k} we have

$$\begin{aligned}\nabla^p_{1,\widetilde{e}_i} = \nabla_{\widetilde{e}_i} + (\Gamma^{\Lambda^{0,\bullet},LC} + \Gamma^\xi + p\Gamma^L)(\widetilde{e}_i) + \frac{1}{\sqrt{2}} S(\widetilde{e}_i, \widetilde{e}_j, f_\alpha) c(\widetilde{e}_j) f^\alpha \\ + \frac{1}{2} S(\widetilde{e}_i, f_\alpha, f_\beta) f^\alpha f^\beta + \frac{1}{2}\left(i_{\widetilde{e}_i}\left(\bar{\partial}^M - \partial^M\right) i\omega\right)^c.\end{aligned} \tag{3.3.7}$$

Let $D^X = \bar{\partial}^X + \bar{\partial}^{X,*}$ be the Dirac operator on $\Lambda^{0,\bullet}(T^*X) \otimes \xi$, and B^ξ the superconnection on B associated with (ω, ξ, h^ξ). Then on U_{x_k}, D^X (resp. B^ξ) can be seen as an operator on $\pi^*\Lambda^\bullet(T^*_\mathbb{R}B)\widehat{\otimes}\Lambda^{0,\bullet}(T^*X) \otimes \xi \otimes L^p$ because the bundle $\pi^*\Lambda^\bullet(T^*_\mathbb{R}B) \otimes L^p$ (resp. L^p) is trivialized. Then, using (3.2.34), [46, Thm. 1.4.7] (which is (3.2.34) in the case where the base B is a point) and (3.3.7), we find that locally,

$$\begin{aligned} B_p^2 &= B^{\xi,2} + p\mathcal{O}_1 + p\mathcal{O}_0^1 + p^2\mathcal{O}_0^2 \\ &= D^{X,2} + R + p\mathcal{O}_1 + p\mathcal{O}_0^1 + p^2\mathcal{O}_0^2 \end{aligned} \tag{3.3.8}$$

where R, $\mathcal{O}_1$ (resp. $\mathcal{O}_0^1$, $\mathcal{O}_0^2$) are operators of order 1 (resp. 0).

Let $s \in \boldsymbol{H}^2(X, \mathbb{E}_p)$, and $s_j = \varphi_j s$, for $1 \leq j \leq N$.

As D^X is elliptic, we have by (3.3.8),

$$\begin{aligned} ||s||_{\boldsymbol{H}^2(p)} &\leq \sum_j C(||D^{X,2}s_j||_{L^2} + ||s_j||_{L^2}) \\ &\leq C\sum_j(||B_p^2 s_j||_{L^2} + p||s_j||_{\boldsymbol{H}^1(p)} + p^2||s_j||_{L^2}). \end{aligned} \tag{3.3.9}$$

Using (3.3.5) and again (3.3.8), we deduce from (3.3.9):

$$||s||_{\boldsymbol{H}^2(p)} \leq C(||B_p^2 s||_{L^2} + p||s||_{\boldsymbol{H}^1(p)} + p^2||s||_{L^2}). \tag{3.3.10}$$

It is easy to see that

$$||s||^2_{\boldsymbol{H}^1(p)} \leq C(||s||_{\boldsymbol{H}^2(p)} + ||s||_{L^2})||s||_{L^2}. \tag{3.3.11}$$

Moreover, if $a, b \in \mathbb{R}_+$ and $\delta > 0$, the Cauchy-Scwarz inequality implies

$$\sqrt{(a+b)b} \leq \delta a + \sqrt{1 + \frac{1}{\delta^2}}\, b. \tag{3.3.12}$$

Thus (3.3.10), (3.3.11) and (3.3.12) with $\delta = (2Cp)^{-1}$ yield to

$$||s||_{\boldsymbol{H}^2(p)} \leq \tilde{C}\left(||B_p^2 s||_{L^2} + p^2||s||_{L^2}\right). \tag{3.3.13}$$

Let Q be a differential operator of order $2m$ with scalar principal symbol and with compact support in U_{x_j}. Then

$$\begin{aligned} \left[B_p^2, Q\right] &= \left[D^{X,2} + R + p\mathcal{O}_1 + p\mathcal{O}_0^1 + p^2\mathcal{O}_0^2, Q\right] \\ &= (\text{order } 2m+1) + (\text{order } 2m)(1+p) + (\text{order } 2m-1)(p+p^2). \end{aligned} \tag{3.3.14}$$

As a consequence, using (3.3.13), we find

$$\begin{aligned} ||Qs||_{\boldsymbol{H}^2(p)} &\leq \tilde{C}\left(||B_p^2 Qs||_{L^2} + p^2||Qs||_{L^2}\right) \\ &\leq C'\left(||QB_p^2 s||_{L^2} + p||s||_{\boldsymbol{H}^{2m+1}(p)} + p^2||s||_{\boldsymbol{H}^{2m-1}(p)} + p^2||Qs||_{L^2}\right) \\ &\leq C''\left(||B_p^2 s||_{\boldsymbol{H}^{2m}(p)} + p^2||s||_{\boldsymbol{H}^{2m}(p)}\right). \end{aligned} \tag{3.3.15}$$

In the last line we use a similar trick as for passing from (3.3.10) to (3.3.13). Hence

$$||s||_{\boldsymbol{H}^{2m+2}(p)} \leq C\left(||B_p^2 s||_{\boldsymbol{H}^{2m}(p)} + p^2||s||_{\boldsymbol{H}^{2m}(p)}\right). \tag{3.3.16}$$

Using (3.3.16), we can finish the proof of Lemma 3.3.1 by induction. □

Let $f\colon \mathbb{R}\to[0,1]$ be a smooth even function such that

$$f(t)=\begin{cases}1 \text{ for } |t|<\varepsilon/2,\\ 0 \text{ for } |t|>\varepsilon.\end{cases} \tag{3.3.17}$$

For $u>0$, $\varsigma\geq 1$ and $a\in\mathbb{C}$, set

$$\begin{aligned}
\mathrm{F}_u(a)&=\int_{\mathbb{R}} e^{iv\sqrt{2}a}\exp(-v^2/2)f(\sqrt{u}v)\frac{dv}{\sqrt{2\pi}},\\
\mathrm{G}_u(a)&=\int_{\mathbb{R}} e^{iv\sqrt{2}a}\exp(-v^2/2)(1-f(\sqrt{u}v))\frac{dv}{\sqrt{2\pi}},\\
\mathrm{H}_{u,\varsigma}(a)&=\int_{\mathbb{R}} e^{iv\sqrt{2}a}\exp(-v^2/2u)(1-f(\sqrt{\varsigma}v))\frac{dv}{\sqrt{2\pi}}.
\end{aligned} \tag{3.3.18}$$

These functions are even holomorphic functions, thus there exist holomorphic functions $\tilde{\mathrm{F}}_u$, $\tilde{\mathrm{G}}_u$ and $\tilde{\mathrm{H}}_{u,\varsigma}$ such that

$$\tilde{\mathrm{F}}_u(a^2)=\mathrm{F}_u(a)\,,\quad \tilde{\mathrm{G}}_u(a^2)=\mathrm{G}_u(a)\quad\text{and}\quad \tilde{\mathrm{H}}_{u,\varsigma}(a^2)=\mathrm{H}_{u,\varsigma}(a). \tag{3.3.19}$$

Moreover, the restriction of $\tilde{\mathrm{F}}_u$ and $\tilde{\mathrm{G}}_u$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$\tilde{\mathrm{G}}_{\frac{u}{p}}\left(\frac{u}{p}a\right)=\tilde{\mathrm{H}}_{\frac{u}{p},1}(a)\quad\text{and}\quad \tilde{\mathrm{F}}_u(vB_p^2)+\tilde{\mathrm{G}}_u(vB_p^2)=\exp\left(-vB_p^2\right)\text{ for } v>0. \tag{3.3.20}$$

Let $\tilde{\mathrm{G}}_u(vB_p^2)(x,x')$ be the smooth kernel of $\tilde{\mathrm{G}}_u(vB_p^2)$ with respect to $dv_X(x')$.

We still denote by π the projection $\pi\colon X\times_B X\to B$ be the projection from the fiberwise product $X\times_B X$ to B. For V, V' two bundle over M, we define the bundle $V\boxtimes V'$ on $X\times_B X$ by

$$(V\boxtimes V')_{(b,x,x')}=V_{(b,x)}\otimes V'_{(b,x')} \tag{3.3.21}$$

for $b\in B$ and $x,x'\in X_b$. Then $\tilde{\mathrm{G}}_u(vB_p^2)(\cdot,\cdot)$ is a section of $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$ over $X\times_B X$. Let $\nabla^{\mathbb{E}_p}$ be the connection on $\mathbb{E}_p$ induced by $\nabla^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}$, $\nabla^{\Lambda^{0,\bullet},LC}$, ∇^L and ∇^ξ, and let $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced connection on $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$. In the same way, let $h^{\mathbb{E}_p}$ be the metric on $\mathbb{E}_p$ induced by $h^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}$, $h^{\Lambda^{0,\bullet},LC}$, h^L and h^ξ, and let $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced metric on $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$.

Proposition 3.3.2. *For any $m\in\mathbb{N}$, $\varepsilon>0$, there exist $C>0$ and $N\in\mathbb{N}$ such that for any $u>0$ and any $p\in\mathbb{N}^*$,*

$$\left|\tilde{\mathrm{G}}_{\frac{u}{p}}\Big(\frac{u}{p}B_p^2\Big)(\cdot\,,\cdot)\right|_{\mathscr{C}^m}\leq Cp^N\exp\left(-\frac{\varepsilon^2p}{16u}\right), \tag{3.3.22}$$

Where the $\mathscr{C}^m$-norm is induced by $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$.

Proof. Observe first that by (3.3.20)

$$\tilde{\mathrm{G}}_{\frac{u}{p}}\left(\frac{u}{p}B_p^2\right)=\tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2). \tag{3.3.23}$$

Moreover, as $i^m a^m e^{iva}=\frac{\partial^m}{\partial v^m}e^{iva}$, we can integrate by part the expression of $a^m\mathrm{H}_{u,\varsigma}(a)$ given in (3.3.18) to obtain that for any $m\in\mathbb{N}$ and $c>0$, there is a $C_{m,c}>0$ such that $u>0$ and $\varsigma\geq 1$,

$$\sup_{|\mathrm{Im}(a)|\leq c}|a^m\mathrm{H}_{u,\varsigma}(a)|\leq C_{m,c}\varsigma^{\frac{m}{2}}\exp\left(-\frac{\varepsilon^2}{16u\varsigma}\right). \tag{3.3.24}$$

For $c>0$, let V_c be the image of $\{a\in\mathbb{C} : |\mathrm{Im}(a)|\leq c\}$ by the map $a\mapsto a^2$. Then

$$V_c=\Big\{a\in\mathbb{C} : \mathrm{Re}(a)\geq \frac{1}{4c^2}\mathrm{Im}(a)^2-c^2\Big\}. \tag{3.3.25}$$

Form (3.3.19) and (3.3.24), we deduce that

$$\sup_{a\in V_c}|a^m\tilde{\mathrm{H}}_{u,\varsigma}(a)|\leq C_{m,c}\varsigma^{\frac{m}{2}}\exp\left(-\frac{\varepsilon^2}{16u\varsigma}\right). \tag{3.3.26}$$

We will prove Proposition 3.3.2 thanks to (3.3.23), (3.3.26) and Lemma 3.3.1. We first need the following lemma.

Lemma 3.3.3. *Let $m\in\mathbb{N}$ and $\phi(a)=a^m\tilde{\mathrm{H}}_{\frac{u}{p},1}(a)$, then there exist $K_m>0$ and an integer $k_m\in\mathbb{N}$ such that*

$$||\phi(B_p^2)s||_{L^2}\leq K_m p^{k_m}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)||s||_{L^2}. \tag{3.3.27}$$

Proof. By Bismut's Lichnerowicz formula (3.2.34), [46, Thm. 1.4.7] and (3.3.7), we have

$$\begin{aligned}&B_p^2=D_p^2+R_p,\\ &R_p\in\mathbb{C}[p]\otimes\Lambda^{\geq 1}(T_{\mathbb{R}}^*B)\otimes \mathrm{Op}_X^{\leq 1}(\Lambda^{0,\bullet}(T^*X)\otimes\xi).\end{aligned} \tag{3.3.28}$$

Where $\mathrm{Op}_X^{\leq 1}(\Lambda^{0,\bullet}(T^*X)\otimes\xi)$ denotes the set of differential operators along the fiber X on $\Lambda^{0,\bullet}(T^*X)\otimes\xi$ of order ≤ 1. We deduce the following fundamental fact:

$$\mathrm{Sp}(B_p^2)=\mathrm{Sp}(D_p^2). \tag{3.3.29}$$

Here, Sp is our notation for the spectrum. Indeed, as R_p has positive degree in $\Lambda^\bullet(T_{\mathbb{R}}^*B)$, we have for $\lambda\notin\mathrm{Sp}(D_p^2)$

$$(\lambda-B_p^2)^{-1}=(\lambda-D_p^2)^{-1}+(\lambda-D_p^2)^{-1}R_p(\lambda-D_p^2)^{-1}+\dots\quad\text{(finite sum)}. \tag{3.3.30}$$

Now, $(\lambda-D_p^2)^{-1}$ is elliptic of order 2, so increases the Sobolev regularity by 2, and R_p is of order 1, thus $(\lambda-B_p^2)^{-1}$ is a bounded operator when acting on the Sobolev space of order 0. This proves that $\lambda\notin\mathrm{Sp}(B_p^2)$. Exchanging the role of B_p^2 and D_p^2, we also prove that if $\lambda\notin\mathrm{Sp}(B_p^2)$, then $\lambda\notin\mathrm{Sp}(D_p^2)$, which shows (3.3.29).

By [46, Thm 1.5.8], there exist $C_L>0$ and $\mu_0>0$ such that

$$\mathrm{Sp}(D_p^2)\subset\{0\}\cup]2p\mu_0-C_L,+\infty[. \tag{3.3.31}$$

Let $\mathcal{C}$ be the contour in $\mathbb{C}$ defined by Figure 3.1. By (3.3.25), we can chose c is big enough so that $\mathcal{C}\subset V_c$.

Note that by (3.3.31) and the self-adjointness of D_p^2, there exists $C>0$ such that for $\lambda\in\mathcal{C}$,

$$||(\lambda-D_p^2)^{-1}s||_{L^2}\leq C||s||_{L^2}. \tag{3.3.32}$$

Moreover, for $\lambda\in\mathcal{C}$ and $x\in\mathbb{R}_+$, we have $\frac{x}{|\lambda-x|}\leq\frac{|\lambda|}{|\lambda-x|}+1\leq C|\lambda|$, where C does not depend on $x\in\mathbb{R}_+$. In particular, we have

$$||D_p^2(\lambda-D_p^2)^{-1}s||_{L^2}\leq C|\lambda|||s||_{L^2}. \tag{3.3.33}$$

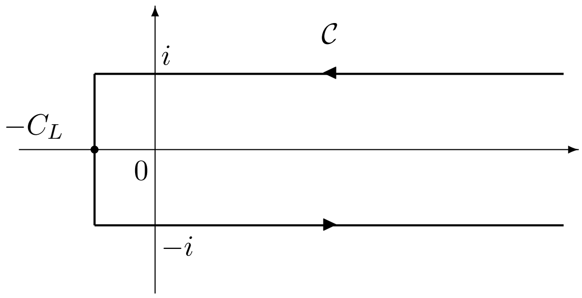

Figure 3.1: The contour $\mathcal{C}$

Now by [46, (1.6.8)] –which is (3.3.13) is the case where B is a point– and by (3.3.31), (3.3.32) and (3.3.33), there is $l \in \mathbb{N}$ and and $C' > 0$ such that for $\lambda \in \mathcal{C}$,

$$\begin{aligned} ||R_p(\lambda - D_p^2)^{-1}s||_{L^2} &\leq Cp^l ||(\lambda - D_p^2)^{-1}s||_{\boldsymbol{H}^2(p)} \\ &\leq Cp^l \left(||D_p^2(\lambda - D_p^2)^{-1}s||_{L^2} + p^2 ||(\lambda - D_p^2)^{-1}s||_{L^2} \right) \\ &\leq C'|\lambda| p^{l+2} ||s||_{L^2}. \end{aligned} \tag{3.3.34}$$

Thus, by (3.3.30), and (3.3.34), we find

$$||(\lambda - B_p^2)^{-1}s||_{L^2} \leq C|\lambda|^k p^{k'} ||s||_{L^2}. \tag{3.3.35}$$

By (3.3.26), we have $|\phi(\lambda)| \leq C_{m+k+2,c} \exp\left(-\frac{\varepsilon^2 p}{16u}\right) |\lambda|^{-(k+2)}$ for $\lambda \in \mathcal{C} \subset V_c$. Using this fact, (3.3.35) and the integral representation

$$\phi(B_p^2) = \frac{1}{2i\pi} \int_{\mathcal{C}} \phi(\lambda)(\lambda - B_p^2)^{-1} d\lambda, \tag{3.3.36}$$

we get Lemma 3.3.3. □

Let Q be a differential operator of order $2m$, $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_{x_i}. Observe that Lemmas 3.3.1 and 3.3.3 are still true if we replace B_p therein by B_p^*, because $B_p^{*,2}$ has the same structure as in (3.3.8) and is equal to $D_p^2 + R_p^*$. Thus, using Lemmas 3.3.1 and 3.3.3, we find that for $m' \in 2\mathbb{N}$,

$$\begin{aligned} \left| \langle B_p^{m'} \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2) Q s, s' \rangle \right| &= \left| \langle s, Q^* \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^{*,2}) B_p^{*,m'} s' \rangle \right| \\ &\leq C\|s\|_{L^2} \left\| \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^{*,2}) B_p^{*,m'} s' \right\|_{\boldsymbol{H}^{2m}(p)} \\ &\leq C\|s\|_{L^2} p^{4m} \sum_{j=0}^{m} p^{-4j} \left\| B_p^{*,2j} \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^{*,2}) B_p^{*,m'} s' \right\|_{L^2} \\ &\leq CKp^{4m+k_m} \exp\left(-\frac{\varepsilon^2 p}{16u}\right) \|s\|_{L^2} \|s'\|_{L^2}. \end{aligned} \tag{3.3.37}$$

Thus,

$$\left\| B_p^{m'} \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2) Q s \right\|_{L^2} \leq CKp^{4m+k_m} \exp\left(-\frac{\varepsilon^2 p}{16u}\right) \|s\|_{L^2}. \tag{3.3.38}$$

We deduce from this estimate – and using once again Lemmas 3.3.1 and 3.3.3 – that if P, Q are differential operators of order $2m', 2m$ respectively and with compact support in U_{x_i}, U_{x_j} respectively, then there is a positive constant $C_{m,m'}$ such that

$$\left\| P\tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2)Qs\right\|_{L^2} \leq C_{m,m'}p^{4m+k_m}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)\|s\|_{L^2}. \tag{3.3.39}$$

By the Sobolev inequality and (3.3.39), we get

$$\left|\tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2)(\cdot,\cdot)\right|_{\mathscr{C}^m(X\times X)} \leq Cp^N\exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{3.3.40}$$

With this estimate and (3.3.23), we get (3.3.22) for the $\mathscr{C}^m$-norm in the directions of the fiber X.

We now turn to the derivatives in the directions of the base B.

Let $k \in \mathbb{N}$. Using (3.3.26) (see [7, (11.57)]), we see that there is a unique holomorphic function $\tilde{\mathrm{H}}_{u,\varsigma,k}$ defined on a neighborhood of V_c such that

$$\frac{\tilde{\mathrm{H}}^{(k-1)}_{u,\varsigma,k}(a)}{(k-1)!} = \tilde{\mathrm{H}}_{u,\varsigma}(a) \tag{3.3.41}$$

and for $u > 0$ and $\varsigma \geq 1$,

$$\sup_{a\in V_c}|a^m\tilde{\mathrm{H}}_{u,\varsigma,k}(a)| \leq C\varsigma^{\frac{m}{2}}\exp\left(-\frac{\varepsilon^2}{16u\varsigma}\right). \tag{3.3.42}$$

For any $q, k \in \mathbb{N}$ and $U \in T_{\mathbb{R}}B$, we have

$$(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^q\tilde{\mathrm{G}}_{\frac{u}{p}}\Big(\frac{u}{p}B_p^2\Big) = \frac{1}{2i\pi}\int_{\mathcal{C}}\tilde{\mathrm{H}}_{\frac{u}{p},1,k}(\lambda)(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^q(\lambda-B_p^2)^{-k}d\lambda, \tag{3.3.43}$$

where U^H denotes the horizontal lift of U in $T^H_{B,\mathbb{R}}M$.

We now prove the analogue of Lemma 3.3.3 for $(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^q\tilde{\mathrm{G}}_{\frac{u}{p}}\big(\frac{u}{p}B_p^2\big)$:

Lemma 3.3.4. *Let $q, m, m' \in \mathbb{N}$. There exist $K_{q,m,m'} > 0$ and an integer $k_{q,m,m'} \in \mathbb{N}$ such that*

$$\left\|B_p^{2m}(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^q\tilde{\mathrm{G}}_{\frac{u}{p}}\Big(\frac{u}{p}B_p^2\Big)B_p^{2m'}s\right\|_{L^2} \leq K_{q,m,m'}p^{k_{q,m,m'}}\exp\left(-\frac{\varepsilon^2 p}{16u}\right)||s||_{L^2}. \tag{3.3.44}$$

Proof. We choose $k \in \mathbb{N}$ so that $k \geq 2(m+m')+q+1$. Then $B_p^{2m}(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^q(\lambda-B_p^2)^{-k}B_p^{2m'}$ can be written as a sum of terms

$$A_1(\lambda-B_p^2)^{-1}A_2(\lambda-B_p^2)^{-1}\dots A_{k+1}(\lambda-B_p^2)^{-1}, \tag{3.3.45}$$

where

$$A_i \in \Big\{1, B_p, (\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^{q'}B_p^2, \Big[B_p, (\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*})^{q'}B_p^2\Big] \,:\, 0\leq q'\leq q\Big\}. \tag{3.3.46}$$

In any case, A_i is a polynomial in p with values in the differential operators along the fiber of order less than 2 (for the last type of term in the above list, we use that B_p is of order 1 and that B_p^2 as a scalar principal symbol). As a consequence,

$$\begin{aligned}||A_i(\lambda-B_p^2)^{-1}s||_{L^2} &\leq Cp^l||(\lambda-B_p^2)^{-1}s||_{\boldsymbol{H}^2(p)}\\ &\leq Cp^l\left(||B_p^2(\lambda-B_p^2)^{-1}s||_{L^2}+p^2||(\lambda-B_p^2)^{-1}s||_{L^2}\right)\\ &\leq Cp^l\left(||(D_p^2+R_p)(\lambda-B_p^2)^{-1}s||_{L^2}+p^2||(\lambda-B_p^2)^{-1}s||_{L^2}\right).\end{aligned} \tag{3.3.47}$$

Using (3.3.30), (3.3.33), (3.3.34) and (3.3.35) we find

$$||A_i(\lambda - B_p^2)^{-1}s||_{L^2} \leq C|\lambda|^a p^b ||s||_{L^2}. \tag{3.3.48}$$

By the decomposition indicated in the begging of the proof, this yields

$$\left\| B_p^{2m}(\nabla_{U^H}^{\mathbb{E}_p \boxtimes \mathbb{E}_p^*})^q(\lambda - B_p^2)^{-k}B_p^{2m'}s\right\|_{L^2} \leq C|\lambda|^c p^d ||s||_{L^2}. \tag{3.3.49}$$

From (3.3.42), (3.3.43) and (3.3.49), we deduce Lemma 3.3.4. □

Using Lemma 3.3.4 in the same way as we used Lemma 3.3.3 to prove (3.3.39) and (3.3.40), we find

$$\left|(\nabla_{U^H}^{\mathbb{E}_p \boxtimes \mathbb{E}_p^*})^q \tilde{\mathrm{H}}_{\frac{u}{p},1}(B_p^2)(\cdot\,,\cdot)\right|_{\mathscr{C}^m(X\times X)} \leq Cp^N \exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{3.3.50}$$

Which completes the proof of Proposition 3.3.2. □

Corollary 3.3.5. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, there exist $C(u) > 0$ a rational fraction in $\sqrt{u}$ and $N \in \mathbb{N}$ such that for any $u > 0$ and any $p \in \mathbb{N}^*$,*

$$\left|\psi_{1/\sqrt{p}}\tilde{\mathrm{G}}_{\frac{u}{p}}(B_{p,u/p}^2)(\cdot,\cdot)\right|_{\mathscr{C}^m} \leq C(u)p^N \exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{3.3.51}$$

Proof. As $B_{p,u} = \sqrt{u}\psi_{1/\sqrt{u}}B_p\psi_{\sqrt{u}}$, we have

$$\psi_{1/\sqrt{p}}\tilde{\mathrm{G}}_{\frac{u}{p}}(B_{p,u/p}^2)\psi_{\sqrt{p}} = \psi_{1/\sqrt{u}}\tilde{\mathrm{G}}_{\frac{u}{p}}\left(\frac{u}{p}B_p^2\right)\psi_{\sqrt{u}}. \tag{3.3.52}$$

Thus, Corollary 3.3.5 follows from Proposition 3.3.2. □

3.3.2 Rescaling B_p

Fix $b_0 \in B$ and $x_0 \in X_{b_0}$. In this section, we will again work along the fiber X_{b_0}, which will be again denoted simply by X. For the rest of this section, we fix $\{w_j\}$ an orthonormal basis of $T_{x_0}^{(1,0)}X$, with dual basis $\{w^j\}$, and we construct an orthonormal basis $\{e_i\}$ of $T_{\mathbb{R},x_0}X$ from $\{w_j\}$ as follows:

$$e_{2j-1} = \frac{1}{\sqrt{2}}\left(w_j + \overline{w}_j\right) \text{ and } e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}\left(w_j - \overline{w}_j\right), \text{ for } 1 \leq j \leq n. \tag{3.3.53}$$

For $\varepsilon > 0$ small enough, we identify $B^{T_{\mathbb{R},x_0}X}(0,\varepsilon)$ and $B^X(x_0,\varepsilon)$ as in Section 3.3.1. Note that in this identification, the radial vector field $\mathcal{R} = \sum_i Z_i e_i$ becomes $\mathcal{R} = Z$, so Z can be viewed as a point or as a tangent vector.

Recall that $\nabla_1^p = \nabla_1 \otimes 1 + 1 \otimes \nabla^{L^p}$ is the connection on $\Lambda^\bullet(T_{\mathbb{R}}^*B) \otimes \Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes \xi$ corresponding to ∇_u in (3.2.33), replacing ξ by $\xi \otimes L^p$ and taking $u = 1$.

For $Z \in B^{T_{\mathbb{R},x_0}X}(0,\varepsilon)$, we identify $(\Lambda_Z^{0,\bullet}(T^*X)\otimes\xi_Z, h_Z^{\Lambda^{0,\bullet}\otimes\xi})$ with $(\Lambda_{x_0}^{0,\bullet}(T^*X)\otimes\xi_{x_0}, h_{x_0}^E)$ and (L_Z, h_Z^L) with $(L_{x_0}, h_{x_0}^L)$ by parallel transport along the geodesic ray $t \in [0,1] \mapsto tZ$ with respect to the connection ∇_1 and ∇^L respectively. We denote by Γ_1 and Γ^L the corresponding connection forms.

We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_0}X$.

Let $\rho\colon \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = \begin{cases} 1 \text{ for } |v| < 2, \\ 0 \text{ for } |v| > 4. \end{cases} \tag{3.3.54}$$

On the trivial bundle

$$\mathbb{E}_{p,x_0} = \Lambda^{\bullet}(T^*_{\mathbb{R},b_0}B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi \otimes L^p\right)_{x_0} \tag{3.3.55}$$

over $T_{x_0}X$, we define the connection

$$\nabla^{\mathbb{E}_{p,x_0}} = \nabla + \rho(|Z|/\varepsilon)\left(p\Gamma^L + \Gamma_1\right), \tag{3.3.56}$$

which is a Hermitian connection Let $g^{T_\mathbb{R}X_0}$ be a Riemannian metric on $X_0 := T_{\mathbb{R},x_0}X = \mathbb{R}^{2n}$ such that

$$g^{T_\mathbb{R}X_0} = \begin{cases} g^{T_\mathbb{R}X} \text{ on } B^{T_{\mathbb{R},x_0}X}(0,2\varepsilon), \\ g^{T_{\mathbb{R},x_0}X} \text{ outside of } B^{T_{\mathbb{R},x_0}X}(0,4\varepsilon), \end{cases} \tag{3.3.57}$$

and let dv_{X_0} be the associated volume form.

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$, and $\kappa(Z)$ be the smooth positive function defined by

$$dv_{X_0}(Z) = \kappa(Z)dv_{TX}(Z), \tag{3.3.58}$$

with $\kappa(0) = 1$.

Let $\Delta^{\mathbb{E}_{p,x_0}}$ be the Bochner Laplacian associated with $\nabla^{\mathbb{E}_{p,x_0}}$ and $g^{T_\mathbb{R}X_0}$. By definition, if $\nabla^{T_\mathbb{R}X_0,LC}$ is the Levi-Civita connection on $(X_0, g^{T_\mathbb{R}X_0})$ and if $(g^{ij}(Z))$ is the inverse of the matrix $(g_{ij}(Z)) = (g_Z^{T_\mathbb{R}X_0}(e_i,e_j))$, we have

$$\Delta^{\mathbb{E}_{p,x_0}} = -g^{ij}(Z)\left(\nabla^{\mathbb{E}_{p,x_0}}_{e_i}\nabla^{\mathbb{E}_{p,x_0}}_{e_j} - \nabla^{\mathbb{E}_{p,x_0}}_{\nabla^{T_\mathbb{R}X_0,LC}_{e_i}e_j}\right). \tag{3.3.59}$$

Recall that $\{f_\alpha\}$ denotes a frame of $T_\mathbb{R}B$, with dual frame $\{f^\alpha\}$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to $\nabla^{T_\mathbb{R}X_0,LC}$ along the curve $t \in [0,1] \mapsto tZ$. Then $\{\widetilde{e}_i\}_i$ is an orthonormal frame of $T_\mathbb{R}X_0$.

Set

$$\begin{aligned}\Phi = \frac{K^X}{8} + \frac{1}{4}c(\widetilde{e}_i)c(\widetilde{e}_j)L'^{\xi}(\widetilde{e}_i,\widetilde{e}_j) + \frac{1}{\sqrt{2}}c(\widetilde{e}_i)f^\alpha L'^{\xi}(\widetilde{e}_i,f_\alpha) + \frac{f^\alpha f^\beta}{2}L'^{\xi}(f_\alpha,f_\beta) \\ - \left(\bar\partial^M\partial^M i\omega\right)^c - \frac{1}{16}\left\| \left(\bar\partial^X - \partial^X\right) i\omega^X\right\|^2_{\Lambda^\bullet(T^*_\mathbb{R}X)}\end{aligned} \tag{3.3.60}$$

and

$$\begin{aligned}M_{p,x_0} = \frac{1}{2}\Delta^{\mathbb{E}_{p,x_0}} + \rho(|Z|/\varepsilon)\Phi \\ + p\rho(|Z|/\varepsilon)\left(\frac{1}{4}c(\widetilde{e}_i)c(\widetilde{e}_j)R^L(\widetilde{e}_i,\widetilde{e}_j) + \frac{1}{\sqrt{2}}c(\widetilde{e}_i)f^\alpha R^L(\widetilde{e}_i,f_\alpha) + \frac{f^\alpha f^\beta}{2}R^L(f_\alpha,f_\beta)\right).\end{aligned} \tag{3.3.61}$$

Then M_{p,x_0} is a second order elliptic differential operator acting on $\mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{p,x_0})$. Moreover, using Theorem 3.2.8, (3.3.56), (3.3.59), (3.3.60) and (3.3.61), we see that M_{p,x_0} and B_p^2 coincide over $B^{TX}(0,2\varepsilon)$.

Let S_L be a unit vector of L_{x_0}. It gives an isometry $L^p_{x_0} \simeq \mathbb{C}$, which yields to an isometry

$$\mathbb{E}_{p,x_0} \simeq \Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes (\Lambda^{0,\bullet}(T^*X) \otimes \xi)_{x_0} =: \mathbb{E}_{x_0}. \tag{3.3.62}$$

We endow $\mathbb{E}$ with the connection $\nabla^{\mathbb{E}}$ induce by $\nabla^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}$, $\nabla^{\Lambda^{0,\bullet},LC}$ and ∇^ξ and with the metric $h^{\mathbb{E}}$ induce by $h^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}$, $h^{\Lambda^{0,\bullet},LC}$ and h^ξ.

Remark 3.3.6. In this trivialization, B_p^2 acts on $\mathbb{E}_{x_0}$, but this action may *a priori* depends on the choice of S_L. However, thanks to Theorem 3.2.8 we see that the operator B_p^2 has it coefficients in $\mathrm{End}(\mathbb{E}_{p,x_0})$ which is canonically isomorphic to $\mathrm{End}(\mathbb{E})_{x_0}$ (by the natural identification $\mathrm{End}(L^p) \simeq \mathbb{C}$), thus all our formulas do not depend on this choice. Under this identification, we will consider M_{p,x_0} as an operator acting on $\mathscr{C}^\infty(T_{x_0}X, \mathbb{E}_{x_0})$.

Let $\exp(-B_p^2)(Z,Z')$ and $\exp(-M_{p,x_0})(Z,Z')$ be the smooth heat kernels of B_p^2 and M_{p,x_0} with respect to $dv_{X_0}(Z')$.

Lemma 3.3.7. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $p \in \mathbb{N}^*$,*

$$\left|\exp\Big(-\frac{u}{p}B_p^2\Big)(x_0,x_0) - \exp\Big(-\frac{u}{p}M_{p,x_0}\Big)(0,0)\right|_{\mathscr{C}^m(M)} \leq Cp^N \exp(-\frac{\varepsilon^2 p}{16u}), \tag{3.3.63}$$

where $|\cdot|_{\mathscr{C}^m(M)}$ denotes the $\mathscr{C}^m$-norm in the parameters $b_0 \in B$ and $x_0 \in X$ induced by $\nabla^{\mathrm{End}(\mathbb{E})}$ and $h^{\mathrm{End}(\mathbb{E})}$.

Proof. By (3.3.61), M_{p,x_0} has the same structure as B_p^2. Thus Lemma 3.3.1 and Proposition 3.3.2 are still true if we replace B_p^2 therein by M_{p,x_0}. From the fact that M_{p,x_0} and B_p^2 coincide near 0 and the finite propagation speed of the wave equation (see e.g. [46, Thm. D.2.1]), we know that

$$\tilde{\mathrm{F}}_{\frac{u}{p}}\left(\frac{u}{p}B_p^2\right)(x_0,\cdot) = \tilde{\mathrm{F}}_{\frac{u}{p}}\left(\frac{u}{p}M_{p,x_0}\right)(0,\cdot), \tag{3.3.64}$$

so we get our Lemma by (3.3.20). □

We will now make the change of parameter $t = \frac{1}{\sqrt{p}} \in]0,1]$.

Definition 3.3.8. For $s \in \mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$ and $Z \in \mathbb{R}^{2n}$ set

$$\begin{aligned}
&(S_t s)(Z) = s(Z/t),\\
&\nabla_t = tS_t^{-1}\kappa^{1/2}\nabla^{\mathbb{E}_{p,x_0}}\kappa^{-1/2}S_t,\\
&\nabla_0 = \nabla + \frac{1}{2}R^L_{x_0}(Z,\cdot),\\
&\mathscr{L}_t = t^2 S_t^{-1}\kappa^{1/2}M_{p,x_0}\kappa^{-1/2}S_t,\\
&\mathscr{L}_0 = -\frac{1}{2}\sum_i(\nabla_{0,e_i})^2 + \frac{1}{4}c(e_i)c(e_j)R^L_{i,j}(x_0) + \frac{1}{\sqrt{2}}c(e_i)f^\alpha R^L_{i,\alpha}(x_0) + \frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta}(x_0).
\end{aligned} \tag{3.3.65}$$

From now on we will denote $c(e_i)$ by c^i to simplify the notation in the computations.

Proposition 3.3.9. *When $t \to 0$, we have*

$$\nabla_{t,e_i} = \nabla_{0,e_i} + O(t) \ \textit{and} \quad \mathscr{L}_t = \mathscr{L}_0 + O(t). \tag{3.3.66}$$

Proof. By (3.3.56) and (3.3.65), we have

$$\nabla_{t,e_i}(Z) = \kappa^{1/2}(tZ)\left\{\nabla_{e_i} + \rho(t|Z|/\varepsilon)\left(t^{-1}\Gamma^L_{tZ}(e_i) + t\Gamma_{1,tZ}(e_i)\right)\right\}\kappa^{-1/2}(tZ). \quad (3.3.67)$$

It is a well known fact (see for instance [46, Lemma 1.2.4]) that for if $\Gamma = \Gamma^L$ (resp. Γ_1) and $R = R^L$ (resp. R_1 the curvature of ∇_1), then

$$\Gamma_Z(e_i) = \frac{1}{2}R_{x_0}(Z, e_i) + O(|Z|^2). \quad (3.3.68)$$

Thus,

$$\begin{aligned} t\Gamma_{1,tZ}(e_i) &= O\left(t^2\right), \\ t^{-1}\Gamma^L_{tZ}(e_i) &= \frac{1}{2}R^L_{x_0}(Z, e_i) + O(t). \end{aligned} \quad (3.3.69)$$

The first asymptotic development in Proposition 3.3.9 follows from $\rho(0) = \kappa(0) = 1$, (3.3.67), (3.3.68) and (3.3.69). Moreover, with this asymptotic, (3.3.59) and the fact that $g^{ij}(0) = \delta_{ij}$ we find

$$\begin{aligned} t^2 S_t^{-1}\kappa^{1/2}\Delta^{\mathbb{E}_{p,x_0}}\kappa^{-1/2}S_t &= -g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TX_0}_{e_i}e_j}\right) \\ &= \sum_i (\nabla_{0,e_i})^2 + O(t). \end{aligned} \quad (3.3.70)$$

Set

$$A_p = M_{p,x_0} - \frac{1}{2}\Delta^{\mathbb{E}_{p,x_0}}. \quad (3.3.71)$$

By (3.3.61) and (3.3.71), we have

$$\begin{aligned} t^2 S_t^{-1} A_p S_t \\ &= \rho(t|Z|/\varepsilon))\left\{\kappa^{1/2}\left(t^2\Phi + \frac{1}{4}c(\widetilde{e_i})c(\widetilde{e_j})R^L(\widetilde{e_i},\widetilde{e_j}) + \frac{1}{\sqrt{2}}c(\widetilde{e_i})f^\alpha R^L(\widetilde{e_i}, f_\alpha)\right.\right. \\ &\qquad \left.\left. + \frac{f^\alpha f^\beta}{2}R^L(f_\alpha, f_\beta)\right)\kappa^{-1/2}\right\}_{tZ} \\ &= \frac{1}{4}c^i c^j R^L_{i,j}(x_0) + \frac{1}{\sqrt{2}}c^i f^\alpha R^L_{i,\alpha}(x_0) + \frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta}(x_0) + O(t). \end{aligned} \quad (3.3.72)$$

With (3.3.65), (3.3.70), (3.3.71), (3.3.72), and the first part of (3.3.66) that we have already proved, the proof of the proposition is completed. □

3.3.3 Convergence of the heat kernel

In this section, we use the notations and definitions of Section 3.3.2. In particular, $b_0 \in B$ and $x_0 \in X_{x_0}$ are fixed.

Definition 3.3.10. Set

$$\Omega_u = uR^L(w_k, \overline{w}_\ell)\overline{w}^\ell \wedge i_{\overline{w}_k} + \sqrt{\frac{u}{2}}c(e_i)f^\alpha R^L_{i,\alpha} + \frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta}. \quad (3.3.73)$$

The purpose of this section is to prove the following result:

Theorem 3.3.11. *Let $k \in \mathbb{N}$. Then there is a $\epsilon > 0$ such that as $p \to +\infty$, uniformly as u varies in a compact subset of $\mathbb{R}_+^*$, we have the following asymptotic for the $\mathscr{C}^k$-norm on $\mathscr{C}^\infty(M, \mathrm{End}(\mathbb{E}))$:*

$$\psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})(x_0, x_0) = \\ (2\pi)^{-n} \exp(-\Omega_{u,x_0}) \frac{\det(\dot{R}^{X,L}_{x_0})}{\det\left(1 - \exp(-u\dot{R}^{X,L}_{x_0})\right)} \otimes \mathrm{Id}_\xi\, p^n + O(p^{n-\epsilon}). \quad (3.3.74)$$

To prove this theorem, we will adapt the method of [46, Sect. 1.6].

Remark 3.3.12. In the proof of this Theorem, we do not use the positivity assumption on L. In this case, we have to us [46, (E.2.5)] in addition to [46, (E.2.4)] to get (3.3.125), and we have to take the convention that if an eigenvalue of $\dot{R}^{X,L}_{x_0}$ is zero, then its contribution to $\frac{\det(\dot{R}^{X,L}_{x_0})}{\det\left(1-\exp(-u\dot{R}^{X,L}_{x_0})\right)}$ is $\frac{1}{2u}$.

Remark 3.3.13. As pointed out in [46, Thm. 4.2.3 and Rem. 4.2.4], we can use the results of this section combined with the techniques of [46, Sect. 4.1] to get $O(p^{n-1})$ instead of $O(p^{n-\epsilon})$ in Theorem 3.3.11. However, we do not need this improvement and leave it to the reader.

The following Lemma is an easy consequence of the Arzelà-Ascoli theorem, which we will use several time.

Lemma 3.3.14. *Let Y be a compact manifold and let (E, h^E, ∇^E) be a Hermitian bundle with connection over Y. We can then define, for $k \in \mathbb{N}$, the $\mathscr{C}^k$-norm $|\cdot|_{\mathscr{C}^k}$ on $\mathscr{C}^\infty(Y, E)$. Let $f_n \in \mathscr{C}^\infty(Y, E)$ be a sequence converging weakly to some distribution f. If for any $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$\sup_n |f_n|_{\mathscr{C}^k} \leq C_k, \quad (3.3.75)$$

then f is smooth and f_n converges in the $\mathscr{C}^\infty$ topology to f.

In the sequel, when we add a superscript (0) to the objects introduced above, we mean their part of degree 0 in $\Lambda^\bullet(T^*_{\mathbb{R},b_0}B)$.

Let $||\cdot||_0$ be the L^2 norm on $\mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$ induced by $h^{\Lambda^\bullet(T^*_\mathbb{R}B)}_{x_0}$, $h^{\Lambda^{0,\bullet}}_{x_0}$, $h^\xi_{x_0}$ and the volume form $dv_{TX}(Z)$. For $s \in \mathscr{C}^\infty(X_0, \mathbb{E}_{x_0})$, $m \in \mathbb{N}^*$, and $p \in \mathbb{N}^*$, set

$$\begin{aligned} ||s||^2_{t,0} &= ||s||^2_0, \\ ||s||^2_{t,m} &= \sum_{\ell \leq m} \sum_{i_1,\dots,i_\ell} ||\nabla^{(0)}_{t,e_{i_1}} \cdots \nabla^{(0)}_{t,e_{i_\ell}} s||^2_0, \\ ||s||^2_{0,m} &= \sum_{\ell \leq m} \sum_{i_1,\dots,i_\ell} ||\nabla_{0,e_{i_1}} \cdots \nabla_{0,e_{i_\ell}} s||^2_0. \end{aligned} \quad (3.3.76)$$

We denote by $\boldsymbol{H}^m_t$ the Sobolov space $\boldsymbol{H}^m(X_0, \mathbb{E}_{x_0})$ endowed with the norm $||\cdot||_{t,m}$, and by $\boldsymbol{H}^{-1}_t$ the Sobolev space of order -1 endowed with the norm

$$||s||_{t,-1} = \sup_{s' \in \boldsymbol{H}^1_p \setminus \{0\}} \frac{\langle s, s' \rangle_{t,0}}{||s'||_{t,0}}. \quad (3.3.77)$$

Finally, if $A \in \mathscr{L}(\boldsymbol{H}^k_t, \boldsymbol{H}^m_t)$, we denote by $||A||^{k,m}_t$ the operator norm of A associated with $||\cdot||_{t,k}$ and $||\cdot||_{t,m}$.

Let

$$\mathscr{R}_t = \mathscr{L}_t - \mathscr{L}^{(0)}_t. \quad (3.3.78)$$

Proposition 3.3.15. *There exist constants $C_1, C_2, C_3 > 0$ such that for any $t > 0$ and any $s, s' \in \mathscr{C}^\infty(X_0, \mathbb{E}_{x_0})$,*

$$\begin{aligned} &\langle \mathscr{L}_t^{(0)} s, s \rangle_{t,0} \geq C_1 ||s||_{t,1}^2 - C_2 ||s||_{t,0}^2, \\ &\left| \langle \mathscr{L}_t^{(0)} s, s' \rangle_{t,0} \right| \leq C_3 ||s||_{t,1} ||s'||_{t,1}, \\ &\|\mathscr{R}_t s\|_{t,0} \leq C_4 ||s||_{t,1}. \end{aligned} \tag{3.3.79}$$

Proof. The operators $\nabla_t^{(0)}, \mathscr{L}_t^{(0)}$ are the operators corresponding to $\nabla_t, \mathscr{L}_t$ in the case where B is a point, thus the first two lines of (3.3.79) are proved in [46, Thm. 1.6.7]. We reprove them here for the convenience of the reader. By the first line of (3.3.70) and (3.3.72), we have

$$\langle \mathscr{L}_t^{(0)} s, s \rangle_0 = \frac{1}{2} ||\nabla_t^{(0)} s||_0^2 + \frac{1}{4} \left\langle c^i c^j R_{i,j}^L(x_0) s, s \right\rangle_0 + O\left(t\right) ||s||_{t,0}^2 \tag{3.3.80}$$

which gives the first two estimates of (3.3.79).

By (3.2.33), (3.3.67) and (3.3.69), we have

$$\nabla_{t,e_i} - \nabla_{t,e_i}^{(0)} = \mathscr{O}_0\left(t^2\right), \tag{3.3.81}$$

where by $\mathscr{O}_0(t^\alpha)$ we mean an operator of order 0 which is a $O(t^\alpha)$. Thus, by (3.3.70), (3.3.71), (3.3.72) and (3.3.81), we have

$$\mathscr{R}_t = \nabla_{t,e_i} \mathscr{O}_0\left(t\right) + \mathscr{O}_0(1). \tag{3.3.82}$$

This immediately yields to the last estimate of (3.3.79). □

Let Γ be the contour in $\mathbb{C}$ defined in Figure 3.2.

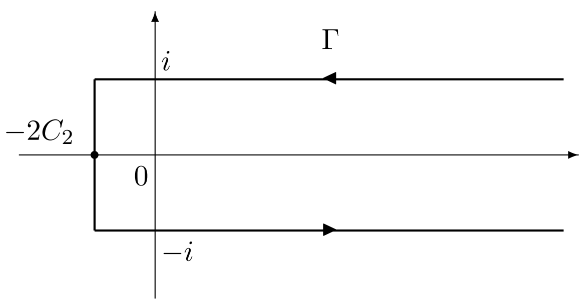

Figure 3.2: The contour Γ

Proposition 3.3.16. *There exist $C > 0$, $a, b \in \mathbb{N}$ such that for any $t > 0$ and any $\lambda \in \Gamma$, the resolvant $(\lambda - \mathscr{L}_t)^{-1}$ exists and*

$$\begin{aligned} &\left\| (\lambda - \mathscr{L}_t)^{-1} \right\|_t^{0,0} \leq C(1 + |\lambda|^2)^a, \\ &\left\| (\lambda - \mathscr{L}_t)^{-1} \right\|_t^{-1,1} \leq C(1 + |\lambda|^2)^b. \end{aligned} \tag{3.3.83}$$

Proof. From (3.3.31), (3.3.65), and the fact that $\mathscr{L}_t^{(0)}$ is self-adjoint, we know that for $\lambda \in \Gamma$, $(\lambda - \mathscr{L}_t^{(0)})^{-1}$ exists and there is a constant $C > 0$ (independent on λ) such that

$$\left\|(\lambda - \mathscr{L}_t^{(0)})^{-1}\right\|_t^{0,0} \leq C. \tag{3.3.84}$$

On the other hand, if $\lambda_0 \in]-\infty, -2C_2]$, then (3.3.79) also implies that

$$\left\|(\lambda_0 - \mathscr{L}_t^{(0)})^{-1}\right\|_t^{-1,1} \leq \frac{1}{C_1}. \tag{3.3.85}$$

Then, using the fact that

$$(\lambda - \mathscr{L}_t^{(0)})^{-1} = (\lambda_0 - \mathscr{L}_t^{(0)})^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t^{(0)})^{-1}(\lambda_0 - \mathscr{L}_t^{(0)})^{-1}, \tag{3.3.86}$$

we find that

$$\left\|(\lambda - \mathscr{L}_t^{(0)})^{-1}\right\|_t^{-1,0} \leq \frac{1}{C_1}\left(1 + C|\lambda - \lambda_0|\right). \tag{3.3.87}$$

Finally, exchanging the last two factors in (3.3.86) and applying (3.3.87), we get

$$\begin{aligned}\left\|(\lambda - \mathscr{L}_t^{(0)})^{-1}\right\|_t^{-1,1} &\leq \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{C_1^2}\left(1 + C|\lambda - \lambda_0|\right) \\ &\leq C(1 + |\lambda|^2).\end{aligned} \tag{3.3.88}$$

For $\lambda \in \Gamma$, we have

$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda - \mathscr{L}_t^{(0)})^{-1} + (\lambda - \mathscr{L}_t^{(0)})^{-1}\mathscr{R}_t(\lambda - \mathscr{L}_t^{(0)})^{-1} + \ldots \quad \text{(finite sum)}. \tag{3.3.89}$$

Moreover, by the third estimate of (3.3.79), we know that

$$\|\mathscr{R}_t\|_t^{1,-1} \leq C_4. \tag{3.3.90}$$

From (3.3.88), (3.3.89) and (3.3.90), we prove (3.3.83). □

Proposition 3.3.17. *Take $m \in \mathbb{N}^*$. Then there exists a contant $C_m > 0$ such that for any $t > 0$, $Q_1, \ldots, Q_m \in \left\{\nabla_{t,e_i}^{(0)}, Z_i\right\}_{i=1}^{2n}$ and $s, s' \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$,*

$$\left|\left\langle [Q_1, [Q_2, \ldots [Q_m, \mathscr{L}_t] \ldots]]s, s'\right\rangle_{t,0}\right| \leq C_m||s||_{t,1}||s'||_{t,1}. \tag{3.3.91}$$

Proof. First, note that $[\nabla_{t,e_i}^{(0)}, Z_j] = \delta_{ij}$. Thus by (3.3.70) and (3.3.72), we know that $[Z_j, \mathscr{L}_t]$ satisfies (3.3.91).

Let $R_{1,\rho}$ and R_ρ^L be the curvatures of the connections $\nabla + \rho(|Z|/\varepsilon)\Gamma_1$ and $\nabla + \rho(|Z|/\varepsilon)\Gamma^L$. Then by (3.3.56) and (3.3.65), we have

$$[\nabla_{t,e_i}^{(0)}, \nabla_{t,e_j}^{(0)}] = (R_\rho^L + t^2 R_{1,\rho})_{tZ}^{(0)}(e_i, e_j). \tag{3.3.92}$$

By (3.3.70), (3.3.72) and (3.3.92), we find that $[\nabla_{t,e_i}^{(0)}, \mathscr{L}_t]$ has the same structure as $\mathscr{L}_t$ for $t \in]0, 1]$, by which we mean that it is of the form

$$\sum_{i,j} a_{ij}(t, tZ)\nabla_{t,e_i}^{(0)}\nabla_{t,e_i}^{(0)} + \sum_i b_i(t, tZ)\nabla_{t,e_i}^{(0)} + c(t, tZ), \tag{3.3.93}$$

where a_{ij}, b_i, c are polynomials in the first variable, and have all their derivatives in the second variable uniformly bounded for $Z \in T_{\mathbb{R},x_0}X$ and $t \in [0,1]$.

The adjoint connection $(\nabla_t^{(0)})^*$ of $\nabla_t^{(0)}$ with respect to $\langle \cdot\,,\cdot\rangle_{t,0}$ is given by

$$(\nabla_t^{(0)})^* = -\nabla_t^{(0)} - t(\kappa^{-1}\nabla\kappa)(tZ). \tag{3.3.94}$$

Note that the last term of (3.3.94) and all its derivative in Z are uniformly bounded for $Z \in T_{\mathbb{R},x_0}X$ and $t \in [0,1]$. Thus, by (3.3.93) and (3.3.94), we find that (3.3.91) holds when $m = 1$.

Finally, we can prove by induction that $[Q_1,[Q_2,\dots[Q_m,\mathscr{L}_t]\dots]]$ has also the same structure as in (3.3.93), and thus satisfies (3.3.91) thanks to (3.3.94). □

Proposition 3.3.18. *For any $t > 0$, $\lambda \in \Gamma$ and $m \in \mathbb{N}$,*

$$(\lambda - \mathscr{L}_t)^{-1}(\boldsymbol{H}_t^m) \subset \boldsymbol{H}_t^{m+1}. \tag{3.3.95}$$

Moreover, for any $\alpha \in \mathbb{N}^{2n}$, there exist $K \in \mathbb{N}$ and $C_{\alpha,m} > 0$ such that for any $t \in]0,1]$, $\lambda \in \Gamma$ and $s \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$,

$$\left\| Z^\alpha(\lambda - \mathscr{L}_t)^{-1}s \right\|_{t,m+1} \leq C_{\alpha,m}(1+|\lambda|^2)^K \sum_{\alpha'\leq\alpha} ||Z^{\alpha'}s||_{t,m}. \tag{3.3.96}$$

Proof. Let $Q_1,\dots,Q_m \in \{\nabla_{t,e_i}^{(0)}\}_{i=1}^{2n}$ and $Q_{m+1},\dots,Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n}$. Then the operator $Q_1\dots Q_{m+|\alpha|}(\lambda-\mathscr{L}_t)^{-1}$ can be decomposed as a linear combination of operators of the type

$$[Q_1,[Q_2,\dots[Q_\ell,(\lambda-\mathscr{L}_t)^{-1}]\dots]]Q_{\ell+1}\dots Q_{m+|\alpha|} \quad \text{with} \quad \ell \leq m+|\alpha|. \tag{3.3.97}$$

We denote by $\mathscr{F}_t$ the family of operator $\mathscr{F}_t = \{[Q_{j_1},[Q_{j_2},\dots[Q_{j_k},\mathscr{L}_t]\dots]]\}$. Then any commutator $[Q_1,[Q_2,\dots[Q_\ell,(\lambda-\mathscr{L}_t)^{-1}]\dots]]$ can be expressed as a linear combination operators of the form

$$(\lambda-\mathscr{L}_t)^{-1}F_1(\lambda-\mathscr{L}_t)^{-1}F_2\dots F_\ell(\lambda-\mathscr{L}_t)^{-1} \quad \text{with} \quad F_j \in \mathscr{F}_t. \tag{3.3.98}$$

Moreover, by Proposition 3.3.17, the norm $\|\cdot\|_t^{1,-1}$ of any element of $\mathscr{F}_t$ is uniformly bounded by C. As a consequence, using Proposition 3.3.16 we see that Proposition 3.3.18 holds. □

Let $e^{-\mathscr{L}_t}(Z,Z')$ be the smooth kernel of the operator $e^{-\mathscr{L}_t}$ with respect to $dv_{TX}(Z')$. Let $\mathrm{pr}_M\colon T_{\mathbb{R}}X \times_M T_{\mathbb{R}}X \to M$ be the projection from the fiberwise product $T_{\mathbb{R}}X \times_M T_{\mathbb{R}}X$ onto M, then $e^{-\mathscr{L}_t}(\cdot,\cdot)$ is a section of $\mathrm{pr}_M^*(\mathrm{End}(\mathbb{E}))$ over $T_{\mathbb{R}}X \times_M T_{\mathbb{R}}X$. Recall that $\nabla^{\mathrm{End}(\mathbb{E})}$ and $h^{\mathrm{End}(\mathbb{E})}$ have been defined below (3.3.62), and let $\nabla^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}$ (resp. $h^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}$) be the induced connection (resp. metric) on $\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})$.

Theorem 3.3.19. *Let $u > 0$ be fixed. For any $m, m' \in \mathbb{N}$, there is $C > 0$ such that for any $t > 0$, $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq 1$,*

$$\sup_{|\alpha|,|\alpha'|\leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} e^{-u\mathscr{L}_t}(Z,Z') \right|_{\mathscr{C}^{m'}(M)} \leq C, \tag{3.3.99}$$

where $|\cdot|_{\mathscr{C}^{m'}(M)}$ denotes the $\mathscr{C}^{m'}$ norm with respect to the parameters b_0 and $x_0 \in X_{b_0}$ induced by $\nabla^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}$ and $h^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}$.

Proof. By (3.3.83), we know that for $k \in \mathbb{N}^*$,

$$e^{-u\mathscr{L}_t} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_\Gamma e^{-u\lambda}(\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{3.3.100}$$

For $m \in \mathbb{N}$, let

$$\mathscr{Q}^m = \left\{ \nabla^{(0)}_{t,e_{i_1}} \cdots \nabla^{(0)}_{t,e_{i_j}} \right\}_{j \le m}. \tag{3.3.101}$$

Then for $m \in \mathbb{N}$, we know from Proposition 3.3.18 that for $Q \in \mathscr{Q}^m$, there are $C_m > 0$ and $M \in \mathbb{N}$ such that for $\lambda \in \Gamma$,

$$\|Q(\lambda - \mathscr{L}_t)^{-m}\|_t^{0,0} \le C_m(1 + |\lambda|^2)^M. \tag{3.3.102}$$

Moreover, if a family $\{Q_t\}$ of differential operators has the structure given in (3.3.93) with $\big(a_{ij}(t,Z)\big)_{ij}$ uniformly positive, then Propositions 3.3.16, 3.3.17 and 3.3.18 also holds for Q_t. By (3.3.70), (3.3.72) and (3.3.94), this is in particular true for $Q_t = \mathscr{L}_t^*$, the formal adjoint of $\mathscr{L}_t$. Hence, for $Q \in \mathscr{Q}^m$,

$$\|Q(\lambda - \mathscr{L}_t^*)^{-m}\|_t^{0,0} \le C_m(1 + |\lambda|^2)^M. \tag{3.3.103}$$

Taking the adjoint of (3.3.103), we deduce

$$\|(\lambda - \mathscr{L}_t)^{-m}Q\|_t^{0,0} \le C_m(1 + |\lambda|^2)^M. \tag{3.3.104}$$

From (3.3.100), (3.3.102) and (3.3.104), we have for $Q, Q' \in \mathscr{Q}^m$:

$$\left\| Q e^{-u\mathscr{L}_t} Q' \right\|_t^{0,0} \le C_m. \tag{3.3.105}$$

Let $\|\cdot\|_m$ be the usual Sobolev norm on $\mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$ induced by $h^{\mathbb{E}_{x_0}}$ and the volume form $dv_{TX}(Z)$:

$$||s||_m^2 = \sum_{\ell \le m} \sum_{i_1,\ldots,i_\ell} ||\nabla_{e_{i_1}} \cdots \nabla_{e_{i_\ell}} s||_0^2. \tag{3.3.106}$$

Then by (3.3.67) and (3.3.76), for any $m \in \mathbb{N}$ there exists $C'_m > 0$ such that for $s \in \mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$ with support in $B^{T_{\mathbb{R},x_0}B}(0,1)$ and $t \in [0,1]$,

$$\frac{1}{C'_m}\|s\|_{t,m} \le \|s\|_m \le C'_m \|s\|_{t,m}. \tag{3.3.107}$$

From (3.3.105), (3.3.107) and Sobolev inequalities (for $\|\cdot\|_m$) we find that (3.3.99) holds when $m' = 0$.

For $m' = 1$, observe that if $U \in T_\mathbb{R}M$, then

$$\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})} e^{-u\mathscr{L}_p} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_\Gamma e^{-\lambda} \nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})} (\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{3.3.108}$$

Moreover, $\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}(\lambda - \mathscr{L}_t)^{-k}$ is a linear combination operators of the form

$$(\lambda - \mathscr{L}_t)^{-i_1}(\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}\mathscr{L}_t)(\lambda - \mathscr{L}_t)^{-i_2} \ldots (\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}\mathscr{L}_t)(\lambda - \mathscr{L}_t)^{-i_\ell}, \tag{3.3.109}$$

and $\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}\mathscr{L}_t$ is a differential operator with the same structure as $\mathscr{L}_t$. In particular, $\nabla_U^{\mathrm{pr}_M^*\mathrm{End}(\mathbb{E})}\mathscr{L}_t$ satisfies an estimates analogous to (3.3.91). Thus, above arguments can be repeated to prove (3.3.99) for $m' = 1$. The case $m' \ge 2$ is similar. □

Theorem 3.3.20. *There are constants $C > 0$ and $M \in \mathbb{N}^*$ such that for $t > 0$,*

$$\left\|((\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1})s\right\|_{0,0} \leq Ct(1 + |\lambda|^2)^M \sum_{|\alpha|\leq 3} ||Z^\alpha s||_{0,0}. \tag{3.3.110}$$

Proof. From (3.3.67) and (3.3.76), for $t \in [0, 1]$ and $m \in \mathbb{N}^*$ we find

$$\|s\|_{t,m} \leq C \sum_{|\alpha|\leq m} \|Z^\alpha s\|_{0,m}. \tag{3.3.111}$$

Moreover, for s, s' with compact support, a Taylor expansion of (3.3.70) and (3.3.72) gives

$$\left|\langle(\mathscr{L}_t - \mathscr{L}_0)s, s'\rangle_{t,0}\right| \leq Ct\|s'\|_{t,1} \sum_{|\alpha|\leq 3} \|Z^\alpha s\|_{0,1}. \tag{3.3.112}$$

Thus,

$$\|(\mathscr{L}_t - \mathscr{L}_0)s\|_{t,-1} \leq Ct \sum_{|\alpha|\leq 3} \|Z^\alpha s\|_{0,1}. \tag{3.3.113}$$

We have

$$(\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1} = (\lambda - \mathscr{L}_t)^{-1}(\mathscr{L}_t - \mathscr{L}_0)(\lambda - \mathscr{L}_0)^{-1}. \tag{3.3.114}$$

Moreover, Propositions 3.3.16, 3.3.17 and 3.3.18 still holds for $t = 0$. Thus, Proposition 3.3.18, (3.3.113) and (3.3.114) yields to (3.3.110). □

Theorem 3.3.21. *For $u > 0$ fixed, there exists $C > 0$ such that for $t > 0$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq 1$,*

$$\left|\left(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0}\right)(Z, Z')\right| \leq Ct^{1/(2n+1)}. \tag{3.3.115}$$

Proof. Let $\mathcal{B} = B^{T_{\mathbb{R},x_0}X}(0, 2)$. Let $\|s\|^2_{\mathcal{B}} = \int_{|Z|\leq 2} |s|^2_{h^{\mathbb{E}_{x_0}}} dv_{TX}(Z)$, and let $J_{x_0} = L^2(\mathcal{B}, \mathbb{E}_{x_0})$. If A is a bounded operator on J_{x_0}, we denote its operator norm by $\|A\|_{\mathcal{B}}$. By (3.3.100) and (3.3.110), we know that there is a $C' > 0$ such that for $t \in]0, 1]$,

$$\begin{aligned} \|e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0}\|_{\mathcal{B}} &\leq \frac{1}{2\pi} \int_\Gamma |e^{-u\lambda}| \|(\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1}\|_{\mathcal{B}} d\lambda \\ &\leq Ct \int_\Gamma e^{-u\mathrm{Re}(\lambda)}(1 + |\lambda|^2)^M d\lambda \leq C't. \end{aligned} \tag{3.3.116}$$

Let $\phi \colon T_{\mathbb{R},x_0}X \to [0, 1]$ be a smooth function with compact support, equal to 1 near 0 and such that $\int_{T_{\mathbb{R},x_0}X} \phi(Z)dv_{TX}(Z) = 1$. Let $\nu \in]0, 1]$. By the proof of Theorem 3.3.19, we see that $e^{-u\mathscr{L}_0}$ satisfies an inequality similar to (3.3.99). By Theorem 3.3.19, there exists $C > 0$ such that for $|Z|, |Z'| \leq 1$ and $U, U' \in \mathbb{E}_{x_0}$,

$$\begin{aligned} &\Big|\Big\langle(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0})(Z, Z')U, U'\Big\rangle \\ &\quad - \int_{T_{\mathbb{R},x_0}X\times T_{\mathbb{R},x_0}X} \Big\langle(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0})(Z - W, Z' - W')U, U'\Big\rangle \\ &\quad\quad \times \frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big| \leq C\nu|U||U'|. \end{aligned} \tag{3.3.117}$$

Moreover, by (3.3.116), we have

$$\begin{aligned} &\Big|\int_{T_{\mathbb{R},x_0}X\times T_{\mathbb{R},x_0}X} \Big\langle(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0})(Z - W, Z' - W')U, U'\Big\rangle \\ &\quad\quad \times \frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big| \leq \frac{Ct}{\nu^{2n}}|U||U'|. \end{aligned} \tag{3.3.118}$$

Hence, we get (3.3.115) by taking $\nu = t^{1/(2n+1)}$. □

We can now prove Theorem 3.3.11.

Let $s \in \mathscr{C}_0^\infty(X_0, \mathbb{E}_{x_0})$. Then by (3.3.58) and (3.3.65)

$$\begin{aligned} e^{-u\mathscr{L}_t} s(Z) &= S_t^{-1}\kappa^{1/2} e^{-\frac{u}{p}M_{p,x_0}}\kappa^{-1/2} S_t(Z) \\ &= \kappa(tZ)\int_{\mathbb{R}^{2n}} e^{-\frac{u}{p}M_{p,x_0}}(tZ,Z')(S_t s)(Z')\kappa^{1/2}(Z')dv_{TX}(Z') \\ &= p^{-n}\kappa(tZ)\int_{\mathbb{R}^{2n}} e^{-\frac{u}{p}M_{p,x_0}}(tZ,tZ'')s(Z'')\kappa^{1/2}(tZ'')dv_{TX}(Z''), \end{aligned} \tag{3.3.119}$$

which yields to

$$e^{-u\mathscr{L}_t}(Z,Z') = p^{-n} e^{-\frac{u}{p}M_{p,x_0}}(tZ,tZ')\kappa^{1/2}(tZ)\kappa^{-1/2}(tZ'). \tag{3.3.120}$$

Define

$$\mathscr{L}_{0,u} = u\psi_{1/\sqrt{u}}\mathscr{L}_0\psi_{\sqrt{u}}. \tag{3.3.121}$$

Then by the last line of (3.3.2), Lemmas 3.3.7 and 3.3.14, Theorem 3.3.19 and 3.3.21 and (3.3.120) we get that for every fixed $u > 0$ and for the $\mathscr{C}^k$-norm on $\mathscr{C}^\infty(M, \mathrm{End}(\mathbb{E}))$,

$$p^{-n}\psi_{1/\sqrt{p}} e^{-B^2_{p,u/p}}(x_0,x_0) = p^{-n}\psi_{1/\sqrt{u}} e^{-\frac{u}{p}M_{p,x_0}}(x_0,x_0) = e^{-\mathscr{L}_{0,u}}(0,0) + O(p^{-\epsilon}), \tag{3.3.122}$$

with $\epsilon = \frac{1}{4n+2}$.

Finally, using the fact that

$$\frac{1}{4}\sum_{ij} c(e_i)c(e_j)R^L(e_i,e_j) = \sum_{l,m} R^L(w_l,\overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l} - \frac{1}{2}\sum_j R^L(w_j,\overline{w}_j) \tag{3.3.123}$$

and (3.1.2), (3.3.65), (3.3.73) and (3.3.121) we find

$$\begin{aligned} \mathscr{L}_{0,u} &= -\frac{u}{2}\sum_i \left(\nabla + \frac{1}{2}R^L_{x_0}(Z,e_i)\right)^2 + u\left(\sum_{l,m} R^L_{x_0}(w_l,\overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l} - \frac{1}{2}\sum_j R^L_{x_0}(w_j,\overline{w}_j)\right) \\ &\quad + \sqrt{\frac{u}{2}} c(e_i) f^\alpha R^L_{i,\alpha}(x_0) + \frac{f^\alpha f^\beta}{2} R^L_{\alpha,\beta}(x_0) \\ &= -\frac{u}{2}\sum_i \left(\nabla + \frac{1}{2}\langle \dot{R}^{X,L}_{x_0} Z, e_i\rangle\right)^2 + \Omega_u(x_0) - \frac{u}{2}\operatorname{Tr}(\dot{R}^{X,L}_{x_0}). \end{aligned} \tag{3.3.124}$$

Hence, the formula for the heat kernel of a harmonic oscillator (see [46, (E.2.4)] for instance) gives

$$e^{-\mathscr{L}_{0,u}}(0,0) = (2\pi)^{-n}\exp(-\Omega_{u,x_0})\frac{\det(\dot{R}^{X,L}_{x_0})}{\det\left(1-\exp(-u\dot{R}^{X,L}_{x_0})\right)} \otimes \mathrm{Id}_\xi, \tag{3.3.125}$$

which implies Theorem 3.3.11 by (3.3.122).

3.3.4 Asymptotic of the torsion forms

Let $b_0 \in B$ be fixed. Again we denote X_{b_0} by X. Recall that ω^H and N_u are defined respectively in (3.2.6) and (3.2.39). let $d = \dim M$.

Definition 3.3.22. For $x \in X$, set

$$\Lambda_u(x) = (2\pi)^{-n} \exp(-\Omega_{u,x}) \frac{\det(\dot{R}^{X,L}_x)}{\det\left(\mathrm{Id} - \exp(-u\dot{R}^{X,L}_x)\right)} \tag{3.3.126}$$

and

$$\mathrm{R}_u(x) = \mathrm{Tr}_s\left[N_u \Lambda_u(x)\right]. \tag{3.3.127}$$

Let $A_j \in \mathscr{C}^\infty(X, \mathrm{End}(\Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \Lambda^{0,\bullet}(T^*X)))$ be such that as $u \to 0$

$$\Lambda_u(x) = \sum_{j=-d}^{k} A_j(x) u^j + O(u^{k+1}). \tag{3.3.128}$$

For convenience, we also set $A_{-d-1} = 0$.

Theorem 3.3.23. *There exist $A_{p,j} \in \mathscr{C}^\infty(X, \Lambda(T^*_{\mathbb{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*X) \otimes \xi))$ such that for any $k, \ell \in \mathbb{N}$, there exist $C > 0$ such that for any $u \in]0,1]$ and $p \geq 1$,*

$$\left| p^{-n} \psi_{1/\sqrt{p}} \exp\left(-B^2_{p,u/p}\right)(x,x) - \sum_{j=-d}^{k} A_{p,j}(x) u^j \right|_{\mathscr{C}^\ell(M)} \leq C u^{k+1}. \tag{3.3.129}$$

Here, $\mathscr{C}^\ell(M)$ denotes the $\mathscr{C}^\ell$-norm in the parameter $(b,x) \in M$.

Moreover, as $p \to +\infty$, we have for any $j \geq -d$

$$A_{p,j}(x) = A_j(x) \otimes \mathrm{Id}_\xi + O\left(\frac{1}{\sqrt{p}}\right), \tag{3.3.130}$$

where the convergence is in the $\mathscr{C}^\infty$ topology on M.

We will prove Theorem 3.3.23 in Section 3.3.5.

For $j \geq -d-1$, set

$$\tilde{A}_j(x) = \mathrm{Tr}_s\left[N_V A_j(x) + i\omega^H A_{j+1}(x)\right]. \tag{3.3.131}$$

Then by (3.2.39), (3.3.127) and (3.3.128), we have

$$\mathrm{R}_u(x) = \sum_{j=-d-1}^{k} \tilde{A}_j(x) u^j + O(u^{k+1}). \tag{3.3.132}$$

Set also

$$\begin{aligned} B_{p,j} &= \int_X \mathrm{Tr}_s\left[N_V A_{p,j}(x) + i\omega^H A_{p,j+1}(x)\right] dv_X(x), \\ B_j &= \int_X \tilde{A}_j(x) dv_X(x). \end{aligned} \tag{3.3.133}$$

Corollary 3.3.24. *For any $k, \ell \in \mathbb{N}$, there exists $C > 0$ such that for any $u \in]0,1]$ and $p \geq 1$,*

$$\left| p^{-n} \psi_{1/\sqrt{p}} \mathrm{Tr}_s\left[N_{u/p} \exp\left(-B^2_{p,u/p}\right)\right] - \sum_{j=-d-1}^{k} B_{p,j} u^j \right|_{\mathscr{C}^\ell(B)} \leq C u^{k+1}. \tag{3.3.134}$$

Moreover, as $p \to +\infty$, we have for any $j \geq -d-1$

$$B_{p,j} = \mathrm{rk}(\xi) B_j + O\left(\frac{1}{\sqrt{p}}\right), \tag{3.3.135}$$

where the convergence is in the $\mathscr{C}^\infty$ topology on B.

Proof. This is a consequence of Theorem 3.3.23, using (3.3.131)-(3.3.133) and $\psi_{1/\sqrt{p}} N_{u/p} = N_u$. □

Theorem 3.3.25. *There exists $C > 0$ such that for $u \geq 1$ and $p \geq 1$,*

$$\left| p^{-n} \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp\left(-B^2_{p,u/p} \right) \right] \right|_{\mathscr{C}^\ell(B)} \leq \frac{C}{\sqrt{u}}. \tag{3.3.136}$$

Theorem 3.3.25 will be proved in Section 3.3.6.

Recall that we assumed in the introduction that there is a $p_0 \in \mathbb{N}$ such that the direct image $R^i \pi_*(\xi \otimes L^p)$ is locally free for all $p \geq p_0$ and $i \in \{1, \ldots, n\}$, and vanishes for $i > 0$. In particular, for $p \geq p_0$,

$$H^i \left(X, (\xi \otimes L^p)|_X \right) = 0 \quad \text{for} \quad i > 0. \tag{3.3.137}$$

For $p \geq p_0$, set

$$\tilde{\zeta}_p(s) = -\frac{p^{-n}}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \psi_{1/\sqrt{p}} \Phi \left\{ \operatorname{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p}) \right] \right\} du. \tag{3.3.138}$$

Here we make an abuse of notation: we should split the integral in two part as in (3.2.48). Clearly, if ζ_p denotes the zeta function (3.2.48) associated with $B_{p,u}$, we have

$$p^{-n} \psi_{1/\sqrt{p}} \zeta_p(s) = p^{-s} \tilde{\zeta}_p(s). \tag{3.3.139}$$

We deduce that

$$p^{-n} \psi_{1/\sqrt{p}} \zeta'_p(0) = \log(p) B_{p,0} + \tilde{\zeta}'_p(0). \tag{3.3.140}$$

On the other hand, we have for $p \geq p_0$,

$$\begin{aligned} \tilde{\zeta}'_p(0) = &- \int_0^1 p^{-n} \Phi \left\{ \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p}) \right] - \sum_{j=-d-1}^{0} B_{p,j} u^j \right\} \frac{du}{u} \\ &- \int_1^{+\infty} p^{-n} \Phi \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p}) \right] \frac{du}{u} - \sum_{j=-d-1}^{-1} \frac{B_{p,j}}{j} + \Gamma'(1) B_{p,0}. \end{aligned} \tag{3.3.141}$$

Let $\tilde{\zeta}(s)$ be the Mellin transform of $u \mapsto -\int_X \mathrm{R}_u(x) dv_X(x)$, i.e., for $\operatorname{Re}(s) > n$:

$$\tilde{\zeta}(s) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} \int_X \mathrm{R}_u(x) dv_X(x) u^{s-1} du. \tag{3.3.142}$$

By Theorem 3.3.23 and Theorem 3.3.25, we can apply the dominated convergence theorem to (3.3.141), and with Theorem 3.3.11 we find

$$\tilde{\zeta}'_p(0) \xrightarrow[p \to +\infty]{} \operatorname{rk}(\xi) \Phi \tilde{\zeta}'(0). \tag{3.3.143}$$

Theorem 3.3.26. *Let $T^{H'}M$ be the orthonormal complement of TX with respect to R^L and let $R^{L,H'} = R^L|_{T^{H'}_{\mathbb{R}} M \times T^{H'}_{\mathbb{R}} M}$. Then*

$$\tilde{\zeta}'(0) = \frac{1}{2} \int_X \det \left(\frac{\dot{R}^{X,L}}{2\pi} \right) \log \left[\det \left(\frac{\dot{R}^{X,L}}{2\pi} \right) \right] e^{-R^{L,H'}} dv_X. \tag{3.3.144}$$

Proof. This Theorem is the analogue of [17, (53)] (see also [46, (5.5.60)]) in the family setting. The main new feature here is the presence in the exponential of terms $c(e_i)f^\alpha$ coupling horizontal and vertical variables. This terms make the computations of the super-traces much more complicated. To deal with them, we inspire us of [52].

We first compute

$$\mathrm{R}_u = (2\pi)^{-n}\,\mathrm{Tr}_s\left(N_u e^{-\Omega_u}\right)\frac{\det(\dot{R}^{X,L})}{\det\left(\mathrm{Id}-\exp(-u\dot{R}^{X,L})\right)}. \tag{3.3.145}$$

Let

$$\widetilde{\Omega}_u = \frac{u}{4}c(e_i)c(e_j)R^L_{ij} + \sqrt{\frac{u}{2}}c(e_i)f^\alpha R^L_{i\alpha}. \tag{3.3.146}$$

Then by (3.3.73) and (3.3.123), we have

$$\Omega_u = \widetilde{\Omega}_u + \frac{f^\alpha f^\beta}{2}R^L_{\alpha\beta} + \frac{u}{2}\,\mathrm{Tr}(\dot{R}^{X,L}), \tag{3.3.147}$$

hence

$$\mathrm{Tr}_s(N_u e^{-\Omega_u}) = \mathrm{Tr}_s(N_u e^{-\widetilde{\Omega}_u})e^{-\frac{f^\alpha f^\beta}{2}R^L_{\alpha\beta}-\frac{u}{2}\mathrm{Tr}(\dot{R}^{X,L})}. \tag{3.3.148}$$

As $c(e_i)c(e_j)\omega_{ij} = 2\sqrt{-1}(\overline{w}^j\wedge i_{\overline{w}_j} - i_{\overline{w}_j}\overline{w}^j)$, we have (see [11, (2.15)])

$$N_V = \frac{n}{2} - \frac{\sqrt{-1}}{4}c(e_i)c(e_j)\omega_{ij} \tag{3.3.149}$$

Recall that ω^X is defined in (3.2.1). Set

$$\begin{aligned} R_u(b) &= -\frac{1}{2}uR^L - \frac{\sqrt{-1}b}{2}\omega^X,\\ \omega_u(b) &= -\widetilde{\Omega}_u - \frac{ib}{2}\omega^Z = \frac{1}{2}c(e_i)c(e_j)R_u(b)_{ij} - \sqrt{\frac{u}{2}}c(e_i)f^\alpha R^L_{i\alpha}. \end{aligned} \tag{3.3.150}$$

Then by (3.2.39), (3.3.149) and (3.3.150) we have

$$\mathrm{Tr}_s(N_u e^{-\widetilde{\Omega}_u}) = \left(\frac{n}{2} + \frac{\sqrt{-1}\omega^H}{u}\right)\mathrm{Tr}_s(e^{\omega_u(0)}) + \left.\frac{\partial}{\partial b}\right|_{b=0}\mathrm{Tr}_s(e^{\omega_u(b)}). \tag{3.3.151}$$

Note that the matrix $\left(R_u(b)_{ij}\right)_{ij}$ is invertible for b small enough. We denote the coefficients of its inverse by $R_u(b)^{ij}$.

Let

$$\begin{aligned} V_i &= \sum_\alpha f^\alpha R^L_{i\alpha}, \quad V_{u,i} = \sqrt{\frac{u}{2}}V_i,\\ \widetilde{V}_{u,i} &= \sum_k R_u(b)^{ik}V_{u,k}. \end{aligned} \tag{3.3.152}$$

A computation shows that

$$\begin{aligned} \omega_u(b) &= \frac{1}{2}c(e_i)c(e_j)R_u(b)_{ij} + V_{u,i}c(e_i)\\ &= \frac{1}{2}\sum_{ij}(c(e_i)-\widetilde{V}_{u,i})R_u(b)_{ij}(c(e_j)-\widetilde{V}_{u,j}) + \frac{1}{2}\sum_{ij}V_{u,i}V_{u,j}R_u(b)^{ij}. \end{aligned} \tag{3.3.153}$$

Hence,

$$\mathrm{Tr}_s(e^{\omega_u(b)}) = \mathrm{Tr}_s\left(e^{\frac{1}{2}(c(e_i)-\widetilde{V}_{u,i})R_u(b)_{ij}(c(e_j)-\widetilde{V}_{u,j})}\right)e^{\frac{1}{2}V_{u,i}V_{u,j}R_u(b)^{ij}}. \tag{3.3.154}$$

Using this equation and [52, Lem. 2.12], we find

$$\operatorname{Tr}_s(e^{\omega_u(b)}) = \operatorname{Tr}_s\left(e^{\frac{1}{2}c(e_i)c(e_j)R_u(b)_{ij}}\right) e^{\frac{1}{2}V_{u,i}V_{u,j}R_u(b)^{ij}}. \tag{3.3.155}$$

We now compute the term $\operatorname{Tr}_s\left(e^{\frac{1}{2}c(e_i)c(e_j)R_u(b)_{ij}}\right)$. We may assume that $\dot{R}^{X,L}$ (see (3.1.2)) is the diagonal matrix $\operatorname{diag}(a_1,\ldots,a_n)$ in the basis $\{w_j\}_j$. Then

$$\begin{aligned}\operatorname{Tr}_s\left(e^{\frac{1}{2}c(e_i)c(e_j)R_u(b)_{ij}}\right) &= \operatorname{Tr}_s\left(\exp\left(-\frac{u}{4}c(e_i)c(e_j)R^L_{ij} - \frac{\sqrt{-1}b}{4}c(e_i)c(e_j)\omega_{ij}\right)\right)\\ &= \operatorname{Tr}_s\left(e^{-u\sum_j a_j\overline{w}^j\wedge i_{\overline{w}_j}+bN_V}\right)e^{\frac{u}{2}\operatorname{Tr}(\dot{R}^{X,L})-\frac{nb}{2}}\\ &= \operatorname{Tr}_s\left(e^{\sum_j(b-u)a_j\overline{w}^j\wedge i_{\overline{w}_j}}\right)e^{\frac{u}{2}\operatorname{Tr}(\dot{R}^{X,L})-\frac{nb}{2}}.\end{aligned} \tag{3.3.156}$$

For $i_1<\cdots<i_k$, we have

$$\left(\sum_j(b-ua_j)\overline{w}^j\wedge i_{\overline{w}_j}\right)\overline{w}^{i_1}\wedge\cdots\wedge\overline{w}^{i_1} = \left(\sum_{\ell=1}^k(b-ua_{i_\ell})\right)\overline{w}^{i_1}\wedge\cdots\wedge\overline{w}^{i_1}. \tag{3.3.157}$$

As a consequence,

$$\operatorname{Tr}_s\left(e^{\sum_j(b-ua_j)\overline{w}^j\wedge i_{\overline{w}_j}}\right) = \sum_{I\subset\{1,\ldots,n\}}(-1)^{|I|}e^{\sum_{i\in I}(b-ua_i)} = \det\left(\operatorname{Id}-e^{b\operatorname{Id}-u\dot{R}^{X,L}}\right), \tag{3.3.158}$$

hence (3.3.155) and (3.3.156) gives

$$\operatorname{Tr}_s(e^{\omega_u(b)}) = \det\left(\operatorname{Id}-e^be^{-u\dot{R}^{X,L}}\right)e^{\operatorname{Tr}(\dot{R}^{X,L})-\frac{nb}{2}}e^{\frac{1}{2}V_{u,i}V_{u,j}R_u(b)^{ij}}. \tag{3.3.159}$$

We now turn to the computation of the derivative at $b=0$ of (3.3.159). Set

$$\begin{aligned}T_{\mathrm{I}} &= \left(\frac{\partial}{\partial b}\Big|_{b=0}\det\left(\operatorname{Id}-e^be^{-u\dot{R}^{X,L}}\right)\right)e^{-\frac{1}{2}V_iV_j(R^L)^{ij}},\\ T_{\mathrm{II}} &= \det\left(\operatorname{Id}-e^be^{-u\dot{R}^{X,L}}\right)\left(\frac{\partial}{\partial b}\Big|_{b=0}e^{\frac{1}{2}V_{u,i}V_{u,j}R_u(b)^{ij}}\right).\end{aligned} \tag{3.3.160}$$

Here $(R^L)^{ij}$ denotes the coefficients of the inverse of the matrix $(R^L_{ij})_{ij}$.

By (3.3.159) we have

$$\frac{\partial}{\partial b}\Big|_{b=0}\operatorname{Tr}_s(e^{\omega_u(b)}) = -\frac{n}{2}\operatorname{Tr}_s(e^{\omega_u(0)}) + (T_{\mathrm{I}}+T_{\mathrm{II}})\,e^{\operatorname{Tr}(\dot{R}^{X,L})}. \tag{3.3.161}$$

First, we get easily

$$T_{\mathrm{I}} = \det\left(\operatorname{Id}-e^{-u\dot{R}^{X,L}}\right)\operatorname{Tr}\left[\left(\operatorname{Id}-e^{u\dot{R}^{X,L}}\right)^{-1}\right]e^{-\frac{1}{2}V_iV_j(R^L)^{ij}}. \tag{3.3.162}$$

Secondly, if $M(b)$ is an invertible matrix, then $(M^{-1})'(b) = -M(b)^{-1}M'(b)M(b)^{-1}$, and

$$\frac{\partial}{\partial b}\Big|_{b=0}e^{\frac{1}{2}{}^tVM(b)^{-1}V} = -\frac{1}{2}{}^tV\left[M(0)^{-1}M'(0)M(0)^{-1}\right]Ve^{\frac{1}{2}{}^tVM(0)^{-1}V}. \tag{3.3.163}$$

Set

$$\left(\omega^X_{R^L}\right)_{ij} = \sum_{kl}(R^L)^{ik}\omega_{kl}(R^L)^{kj} \quad\text{and}\quad \omega^X_{R^L}(V,V) = V_iV_j\left(\omega^X_{R^L}\right)_{ij}. \tag{3.3.164}$$

By (3.3.163), we have

$$T_{\mathrm{II}} = \frac{\sqrt{-1}}{2u} \det\left(\mathrm{Id} - e^{-u\dot{R}^{X,L}}\right) \omega^X_{R^L}(V,V) e^{-\frac{1}{2}V_iV_j(R^L)^{ij}}. \tag{3.3.165}$$

Finally, using (3.3.145), (3.3.148), (3.3.151), (3.3.159), (3.3.161), (3.3.162) and (3.3.165), and defining

$$\mathscr{F}^H = e^{-\frac{1}{2}\left(f^\alpha f^\beta R^L_{\alpha\beta} + V_iV_j(R^L)^{ij}\right)}, \tag{3.3.166}$$

we find

$$\mathrm{R}_u = \left\{ \frac{\sqrt{-1}}{u}\Big(\omega^H + \frac{1}{2}\omega^X_{R^L}(V,V)\Big) + \mathrm{Tr}\left[\left(\mathrm{Id} - e^{u\dot{R}^{X,L}}\right)^{-1}\right] \right\} \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathscr{F}^H. \tag{3.3.167}$$

In the sequel, we will denote with a subscript $\{*\}$ the objects corresponding to the objects defined above in the case where B is a point (e.g. $\mathrm{R}_u^{\{*\}}$, $\tilde{A}_j^{\{*\}}$, ...). This objects are in fact the ones appearing in [17] and [46, Sect. 5.5.4], and are the part of degree 0 of our objects. By (3.3.132), (3.3.167) and [46, (5.5.37)-(5.5.40)] we have

$$\begin{aligned}
&\mathrm{R}_u = \left\{ \frac{\sqrt{-1}}{u}\Big(\omega^H + \frac{1}{2}\omega^X_{R^L}(V,V)\Big) \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) + \mathrm{R}_u^{\{*\}} \right\} \mathscr{F}^H \\
&\tilde{A}_j = \tilde{A}_j^{\{*\}} \mathscr{F}^H \text{ for } j \neq -1\,, \\
&\tilde{A}_{-1} = \left\{ \tilde{A}_{-1}^{\{*\}} + \sqrt{-1}\Big(\omega^H + \frac{1}{2}\omega^X_{R^L}(V,V)\Big) \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \right\} \mathscr{F}^H.
\end{aligned} \tag{3.3.168}$$

In particular,

$$\begin{aligned}
&\tilde{A}_j = 0 \text{ for } j \leq -2, \\
&\mathrm{R}_u - \frac{\tilde{A}_{-1}}{u} - \tilde{A}_0 = \left\{ \mathrm{R}_u^{\{*\}} - \frac{\tilde{A}_{-1}^{\{*\}}}{u} - \tilde{A}_0^{\{*\}} \right\} \mathscr{F}^H.
\end{aligned} \tag{3.3.169}$$

Since $\dot{R}^{X,L} \in \mathrm{End}(T^{(1,0)}X)$ has positive eigenvalues, we find using (3.3.168), (3.3.169) and $\mathrm{R}_u^{\{*\}} = \mathrm{Tr}\left[\left(\mathrm{Id} - e^{u\dot{R}^{X,L}}\right)^{-1}\right] \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathscr{F}^H$ that for $\mathrm{Re}(z) > 1$,

$$\tilde{\zeta}(z) = \left(\int_X \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathrm{Tr}\left[\left(\dot{R}^{X,L}\right)^{-z}\right] \mathscr{F}^H dv_X \right) \frac{1}{\Gamma(z)} \int_0^{+\infty} u^{z-1} \frac{e^{-u}}{1-e^{-u}} du. \tag{3.3.170}$$

Let $\zeta(z) = \sum_{n=0}^{+\infty} \frac{1}{n^z}$ be the Riemann zeta function. Then classically, we have

$$\begin{aligned}
&\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{+\infty} u^{z-1} \frac{e^{-u}}{1-e^{-u}} du, \\
&\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log(2\pi).
\end{aligned} \tag{3.3.171}$$

Finally, (3.3.170) and (3.3.171) yields to

$$\begin{aligned}
\tilde{\zeta}'(0) &= -\zeta(0) \int_X \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathrm{Tr}\left[\log\left(\dot{R}^{X,L}\right)\right] \mathscr{F}^H dv_X + n\zeta'(0) \int_X \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathscr{F}^H dv_X \\
&= \frac{1}{2} \int_X \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \log\left[\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\right] \mathscr{F}^H dv_X.
\end{aligned} \tag{3.3.172}$$

To prove (3.3.144), we now have to prove that $\mathscr{F}^H = e^{-R^{L,H'}}$, i.e.

$$f^\alpha f^\beta R^L_{\alpha\beta} + V_i V_j (R^L)^{ij} = f'^\alpha f'^\beta R^L(f'_\alpha, f'_\beta) \tag{3.3.173}$$

for some basis $\{f'_\alpha\}_\alpha$ of $T^{H'}_{\mathbb{R}} M$ (the right hand side does not depend on the choice of $\{f'_\alpha\}_\alpha$).

We choose f'_α so that $f'_\alpha - f_\alpha = u_\alpha \in T_{\mathbb{R}} X$. Recall that $f^\alpha \in T^*_{\mathbb{R}} M$ is in fact $f^{\alpha,H}$ with $(\cdot)^H \colon T^*_{\mathbb{R}} B \xrightarrow{\sim} T^{H,*}_{\mathbb{R}} M$. On the other hand, if we extend $f'^\alpha \in T^{H',*}_{\mathbb{R}} M$ to $T^*_{\mathbb{R}} M = T^*_{\mathbb{R}} X \oplus T^{H',*}_{\mathbb{R}} M$ in the obvious way. Then we obtain

$$f'^\alpha = f^\alpha \in T^*_{\mathbb{R}} M. \tag{3.3.174}$$

Write $u_\alpha = \sum_i u^i_\alpha e_i$. By (3.3.174), we have on the one hand

$$\begin{aligned} R^L(f'_\alpha, f'_\beta) f'^\alpha f'^\beta &= R^L(f'_\alpha, f_\beta + u^j_\beta e_j) f^\alpha f^\beta \\ &= R^L(f'_\alpha, f_\beta) f^\alpha f^\beta \\ &= \left(R^L_{\alpha\beta} + u^i_\alpha R^L_{i\beta} \right) f^\alpha f^\beta. \end{aligned} \tag{3.3.175}$$

On the other hand,

$$\begin{aligned} R^L_{i,\beta} &= R^L(e_i, f'_\beta - u^k_\beta e_k) \\ &= -u^k_\beta R^L_{ik}, \end{aligned} \tag{3.3.176}$$

so we have by (3.3.152)

$$\begin{aligned} V_i V_j (R^L)^{ij} &= R^L_{i\alpha} R^L_{j\beta} (R^L)^{ij} f^\alpha f^\beta \\ &= u^k_\alpha R^L_{ik} R^L_{j\beta} (R^L)^{ij} f^\alpha f^\beta \\ &= u^j_\alpha R^L_{j\beta} f^\alpha f^\beta. \end{aligned} \tag{3.3.177}$$

By (3.3.175) and (3.3.177), we get (3.3.173). Theorem 3.3.26 is proved. □

We can now finish the proof of Theorem 3.1.3.

Recall that Θ^X is defined in (3.3.1). Then

$$\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) dv_X = \frac{\Theta^{X,n}}{n!}. \tag{3.3.178}$$

By (3.3.167) we have

$$\tilde{A}_0 = \frac{n}{2} \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right) \mathscr{F}^H. \tag{3.3.179}$$

Now by Corollary 3.3.24, (3.3.140), (3.3.143), Theorem 3.3.26, (3.3.178) and (3.3.179), we have in the smooth topology on B as $p \to +\infty$

$$\begin{aligned} \psi_{1/\sqrt{p}} \zeta'_p(0) &= \log(p) p^n B_0 + p^n \Phi \tilde{\zeta}'(0) + o(p^n) \\ &= \frac{\mathrm{rk}(\xi)}{2} \Phi \left\{ \int_X \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right] e^{-R^{L,H'}} \frac{(p\Theta^X)^n}{n!} \right\} + o(p^n) \\ &= \frac{\mathrm{rk}(\xi)}{2} \int_X \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right] \exp\left(\frac{\sqrt{-1}}{2\pi} R^{L,H'} + p\Theta^X\right) + o(p^n), \end{aligned} \tag{3.3.180}$$

which is (3.1.4). Thanks to Corollary 3.3.24, Theorem 3.3.25, (3.3.140)and (3.3.141), we can apply Lemma 3.3.14 to get Theorem 3.1.3.

3.3.5 Proof of Theorem 3.3.23

We follow here the idea of [46, Sect. 5.5.6].

We denote $\mathscr{L}_t$ defined in (3.3.65) by $\mathscr{L}_{t,x_0}$ to make the dependence on x_0 clearer. Then $\mathscr{L}_{t,x_0}$ is a family of differential operators with coefficients in $\mathrm{End}(\mathbb{E}_{x_0})$ which depends on $x_0 \in X$ and $t \in [0,1]$. As explained before Theorem 3.3.19, $e^{-\mathscr{L}_{t,\cdot}}(\cdot,\cdot)$ is a section of $\mathrm{pr}_M^*(\mathrm{End}(\mathbb{E}))$ over $T_{\mathbb{R}}X \times_M T_{\mathbb{R}}X$. Using the techniques of the proof of Proposition 3.3.2 and the fact that for $m \in \mathbb{N}$ and $\varepsilon > 0$ there is a $C_m > 0$ such that for $0 < u \leq 1$

$$\sup_{a\in\Gamma} \left| a^m \tilde{\mathrm{G}}_u(\sqrt{u}a) \right| \leq C_m \exp\Big(-\frac{\varepsilon^2}{16u}\Big), \tag{3.3.181}$$

we get an analogue of Proposition 3.3.2: there is $C > 0$ such that for $u \in]0,1]$,

$$\left| \tilde{\mathrm{G}}_u(u\mathscr{L}_{t,x_0})(0,0) \right|_{\mathscr{C}^m(M\times[0,1])} \leq C \exp\left(-\frac{\varepsilon^2}{32u} \right). \tag{3.3.182}$$

Here the $\mathscr{C}^m$ norm is in the parameters x_0 and t.

By the finite propagation speed of the wave operator [46, Thm D.2.1], for t small, $\tilde{\mathrm{F}}_u(u\mathscr{L}_{t,x_0}(0,\cdot))$ only depend on the restriction of $\mathscr{L}_{t,x_0}$ on $B^{T_{\mathbb{R},x_0}X}(0,2\varepsilon)$ and

$$\mathrm{supp}\left(\tilde{\mathrm{F}}_u(u\mathscr{L}_{t,x_0})(0,\cdot) \right) \subset B^{T_{\mathbb{R},x_0}X}(0,2\varepsilon). \tag{3.3.183}$$

Now consider a sphere bundle $V = \{(z,c) \in T_{\mathbb{R}}X \times \mathbb{R} : |z|^2 + c^2 = 1\}$ over X. We embed $B^{T_{\mathbb{R},x_0}X}(0,2\varepsilon)$ in V_{x_0} by sending z to $(z, \sqrt{1-|z|^2})$ and we extend $\mathscr{L}_{t,x_0}$ to a generalized Laplacian $\widetilde{\mathscr{L}}_{t,x_0}$ on V_{x_0} with values in $\mathrm{pr}_M^*(\mathrm{End}(\mathbb{E}))$. Once again, using $\tilde{G}_u$ as in (3.3.182) and the finite propagation speed as in Lemma 3.3.7, we find

$$\left| e^{-u\mathscr{L}_{t,x_0}}(0,0) - e^{-u\widetilde{\mathscr{L}}_{t,x_0}}(0,0) \right|_{\mathscr{C}^m(M\times[0,1])} \leq C \exp\left(-\frac{\varepsilon^2}{32u} \right). \tag{3.3.184}$$

Finally, as the total space of V is compact, the heat kernel $\exp\left(-u\widetilde{\mathscr{L}}_{t,x_0}\right)(0,0)$ has an asymptotic expansion (starting with u^{-n}) when $u \to 0$ which depends smoothly on the parameters x_0 and t (see for instance [46, (D.1.24)]). Thus Lemma 3.3.7, (3.3.122) and (3.3.184) we find (3.3.129) and $A_{p,j} = A_{\infty,j} + O(1/\sqrt{p})$. Moreover, we get $A_{\infty,j} = A_j \otimes \mathrm{Id}_\xi$ from (3.3.125).

3.3.6 Proof of Theorem 3.3.25

We will use here the notations of Section 3.3.1.

We use the notation of (3.3.28). Let

$$C_p = \frac{1}{p} B_p^2 = \frac{1}{p}(D_p^2 + R_p). \tag{3.3.185}$$

By the last line of (3.3.2), we have

$$p^{-n} \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} e^{-B^2_{p,u/p}} \right] = p^{-n} \operatorname{Tr}_s \left[N_u \psi_{1/\sqrt{u}} e^{-uC_p} \psi_{\sqrt{u}} \right]. \tag{3.3.186}$$

By (3.3.29) and (3.3.31), there exists $\nu > 0$ such that for p large

$$\begin{aligned}&\mathrm{Sp}(D_p/\sqrt{p}) \subset]-\infty, -\sqrt{\nu}] \cup \{0\} \cup [\sqrt{\nu}, +\infty[,\\ &\mathrm{Sp}(C_p) \subset \{0\} \cup [\nu, +\infty[.\end{aligned} \tag{3.3.187}$$

In the sequel, we will assume that (3.3.187) holds for $p \geq 1$. Let δ be the counterclockwise oriented circle in $\mathbb{C}$ centered at 0 and of radius $\nu/2$, and let Δ be the contour in $\mathbb{C}$ defined in Figure 3.3.

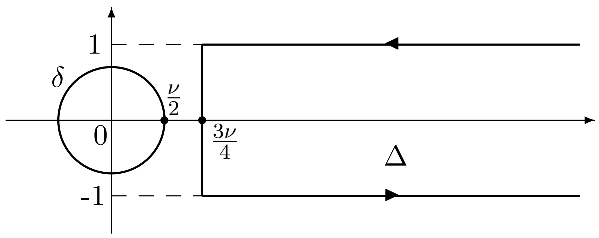

Figure 3.3: The contours δ and Δ

Set

$$\begin{aligned}\mathbb{P}_{p,u} &= \frac{1}{2i\pi}\psi_{1/\sqrt{u}} \int_\delta e^{-u\lambda}(\lambda - C_p)^{-1} d\lambda,\\ \mathbb{K}_{p,u} &= \frac{1}{2i\pi}\psi_{1/\sqrt{u}} \int_\Delta e^{-u\lambda}(\lambda - C_p)^{-1} d\lambda.\end{aligned} \tag{3.3.188}$$

Then

$$p^{-n}\psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} e^{-B^2_{p,u/p}}\right] = p^{-n} \operatorname{Tr}_s \left[N_u(\mathbb{P}_{p,u} + \mathbb{K}_{p,u})\right]. \tag{3.3.189}$$

We will deal separately with the terms $\mathbb{P}_{p,u}$ and $\mathbb{K}_{p,u}$.

In the rest of this section, we will work on a subset of B small enough so that we can assume that $M = B \times X$.

The term involving $\mathbb{K}_{u,p}$

Definition 3.3.27. For $A \in \Lambda^\bullet(T^*_\mathbb{R}B) \otimes \mathrm{End}\left(\Omega^{0,\bullet}(X, \xi \otimes L^p)\right)$, let $||A||_\infty$ be the norm of operator of A viewed as an endomorphism of $L^2(X, \mathbb{E}_p)$ and for $q \in \mathbb{N}^*$, let

$$||A||_q = \left(\operatorname{Tr}\left[(A^*A)^{q/2}\right]\right)^{1/q}. \tag{3.3.190}$$

Note that if $||A||_q$ and $||A'||_\infty$ exist, then

$$||AA'||_q \leq ||A||_q ||A'||_\infty. \tag{3.3.191}$$

Remark 3.3.28. We do not to specify the dependance in $b \in B$ or $p \in \mathbb{N}^*$ of the norm $||\cdot||_q$ to make the notations lighter.

Lemma 3.3.29. *Let $\lambda_0 \in \mathbb{R}^*_-$. Then there exists q_0 such that for $q \geq q_0$, for $U \in T_\mathbb{R}B$ and $\ell \in \mathbb{N}$, there is a $C > 0$ such that for $p \geq 1$*

$$p^{-n}\left\|(\nabla_U^{\mathrm{End}(\mathbb{E}_p)})^\ell (\lambda_0 - C_p)^{-q}\right\|_1 \leq C. \tag{3.3.192}$$

Proof. Set
$$H_p = D_p^2/p - \lambda_0. \tag{3.3.193}$$
Then H_p is a self-adjoint positive generalized Laplacian on X. By [2, Thm. 2.38], we know that for $k > 1 + \frac{\dim_{\mathbb{R}} X + r}{2}$, the operator H_p^{-k} has a $\mathscr{C}^r$ kernel given for $(x,x') \in X \times X$ by
$$H_p^{-k}(x,x') = \frac{1}{(k-1)!}\int_0^{+\infty} e^{-tH_p}(x,x')t^{k-1}dt. \tag{3.3.194}$$
Thus,
$$\begin{aligned}\operatorname{Tr}\left[H_p^{-k}\right] &= \frac{1}{(k-1)!}\int_X\int_0^{+\infty}\operatorname{Tr}\left[e^{-tH_p}(x,x)\right]t^{k-1}dtdv_X(x)\\ &= \frac{1}{(k-1)!}\int_0^{+\infty}\operatorname{Tr}\left[e^{-tH_p}\right]t^{k-1}dt.\end{aligned} \tag{3.3.195}$$

Now, using the degree 0 of Theorem 3.3.11 we find that $p^{-n}\operatorname{Tr}\left[e^{-\frac{1}{p}D_p^2}\right]$ converges (along with its derivatives) when $p \to +\infty$. In particular, $p^{-n}\operatorname{Tr}\left[e^{-\frac{1}{p}D_p^2}\right]$ and its derivative are bounded. Moreover, D_p^2 is a positive operator. Thus, for $\ell \in \mathbb{N}$, there is $C > 0$ such that for $t \geq 1$ and $p \in \mathbb{N}^*$,
$$\begin{aligned}p^{-n}\left|\operatorname{Tr}\left[e^{-tH_p}\right]\right|_{\mathscr{C}^\ell(B)} &= p^{-n}\left|\operatorname{Tr}\left[e^{-\frac{t}{p}D_p^2}\right]\right|_{\mathscr{C}^\ell(B)} e^{\lambda_0 t}\\ &= p^{-n}\left|\operatorname{Tr}\left[e^{-\frac{t-1}{p}D_p^2}e^{-\frac{1}{p}D_p^2}\right]\right|_{\mathscr{C}^\ell(B)} e^{\lambda_0 t}\\ &\leq p^{-n}\left|\operatorname{Tr}\left[e^{-\frac{1}{p}D_p^2}\right]\right|_{\mathscr{C}^\ell(B)} e^{\lambda_0 t} \leq Ce^{\lambda_0 t}.\end{aligned} \tag{3.3.196}$$
Moreover, using the part of degree 0 in Theorem 3.3.23, we find that for any $k, \ell \in \mathbb{N}$, there exist $a_{p,j} \in \mathbb{R}$ and $C > 0$ such that for any $t \in]0,1]$ and $p \geq 1$,
$$\left|p^{-n}\operatorname{Tr}\left[\exp\left(-\frac{t}{p}D_p^2\right)\right] - \sum_{j=-n-1}^{k} a_{p,j}t^j\right|_{\mathscr{C}^\ell(B)} \leq Ct^{k+1}. \tag{3.3.197}$$
To remove the N_V operator in the trace in the above equation, we used that D_p^2 preserves the vertical degree.

Splitting the integral in (3.3.195) at $t = 1$ and using (3.3.196) and (3.3.197), we find
$$p^{-n}\left|\operatorname{Tr}\left[H_p^{-k}\right]\right|_{\mathscr{C}^\ell(B)} \leq C. \tag{3.3.198}$$

Thus, there exists $q_0 \in \mathbb{N}$ such that for $q \geq q_0$ there is $C > 0$ such that
$$p^{-n}\left\|(\lambda_0 - D_p^2/p)^{-q}\right\|_1 = p^{-n}\operatorname{Tr}\left[H_p^{-q}\right] \leq C. \tag{3.3.199}$$

Moreover, by (3.3.187) there is a $C' > 0$ such that for $p \geq 1$,
$$\left\|(\lambda_0 - D_p^2/p)^{-1}\right\|_\infty \leq C'. \tag{3.3.200}$$

A closer look at Bismut's Lichnerowicz formula (3.2.34) and (3.2.35) enables us to sharpen (3.3.28): locally, under the trivialization on U_{x_k} (see Section 3.3.1), we have
$$\frac{1}{p}R_p = \frac{1}{p}\mathcal{O}_1 + \mathcal{O}_0, \tag{3.3.201}$$

were $\mathcal{O}_k$ is a differential operator of order k (which does not depend on p). Moreover, in the same way as in (3.3.10), we can prove

$$||s||_{\boldsymbol{H}^1(p)} \leq C\big(||D_p s||_{L^2} + p||s||_{L^2}\big). \tag{3.3.202}$$

Consequently, if s is an eigenfunction of $D_p/\sqrt{p}$ for the eigenvalue μ,

$$\begin{aligned}\frac{1}{p}||R_p s||_{L^2} &\leq \frac{1}{p}||s||_{\boldsymbol{H}^1(p)} + ||s||_{L^2} \\ &\leq C\frac{1}{p}||D_p s||_{L^2} + C'||s||_{L^2} \\ &\leq C\Big(1+\frac{|\mu|}{\sqrt{p}}\Big)||s||_{L^2} \leq C(1+|\mu|)||s||_{L^2}.\end{aligned} \tag{3.3.203}$$

This estimate yields to

$$\frac{1}{p}\left\|R_p(\lambda_0 - D_p^2/p)^{-1}\right\|_\infty \leq C \sup_{\mu\in[\sqrt{\nu},+\infty[} \frac{1+\mu}{|\lambda_0-\mu^2|} \leq C'. \tag{3.3.204}$$

As in (3.3.30), we have

$$(\lambda_0 - C_p)^{-1} = (\lambda_0 - D_p^2/p)^{-1} + (\lambda_0 - D_p^2/p)^{-1}(R_p/p)(\lambda_0 - D_p^2/p)^{-1} + \,\cdots\,, \tag{3.3.205}$$

with only finitely many terms (as R_p is sum of elements of positive degree in $\Lambda^\bullet(T^*_\mathbb{R}B)$). Thus, for $q\in\mathbb{N}^*$, $(\lambda_0 - C_p)^{-q}$ is a sum of terms of the form

$$(\lambda_0 - D_p^2/p)^{-k_0} R_p/p \cdots R_p/p(\lambda_0 - D_p^2/p)^{-k_i}, \tag{3.3.206}$$

with $0 \leq i \leq \dim_\mathbb{R} B$, $k_j \geq 1$ and $\sum_j k_j = q+i$. In particular, there exist j_0 such that $k_{j_0} \geq \frac{q}{\dim_\mathbb{R} B+1}$. Thus, if q is large enough, then $(\lambda_0 - C_p)^{-q}$ is a sum of product of terms of the form (3.3.206) – which are bounded for $\|\cdot\|_\infty$ by (3.3.200) and (3.3.204) – and of $(\lambda_0 - D_p^2/p)^{-q_0}$. Thus, form (3.3.191) and (3.3.199), we get Lemma 3.3.29 for $\ell = 0$.

Using (3.3.205), we find that $\nabla_U^{\mathrm{End}(\mathbb{E}_p)}(\lambda_0 - C_p)^{-q}$ is a sum of terms

$$(\lambda_0 - D_p^2/p)^{-k_0} A_{k_1}(p)\cdots A_{k_i}(p)(\lambda_0 - D_p^2/p)^{-k_i}, \tag{3.3.207}$$

with $0 \leq i \leq \dim_\mathbb{R} B + 1$, $k_j \geq 1$, $\sum_j k_j = q+i$ and

$$A_{k_j}(p) \in \left\{R_p/p, \nabla_U^{\mathrm{End}(\mathbb{E}_p)} R_p/p, \nabla_U^{\mathrm{End}(\mathbb{E}_p)} D_p^2/p\right\}. \tag{3.3.208}$$

Thus, using the same reasoning as above, to prove Lemma 3.3.29 for $\ell = 1$, we only have to show that there exists $C > 0$ such that for any $p \in \mathbb{N}^*$

$$\left\|A_{k_j}(p)(\lambda_0 - D_p^2/p)^{-1}s\right\|_{L^2} \leq C\|s\|_{L^2}. \tag{3.3.209}$$

By (3.3.204), estimation (3.3.209) holds if $A_{k_j}(p) = R_p/p$. Also, as $\nabla_U^{\mathrm{End}(\mathbb{E}_p)} R_p/p$ has the same structure as R_p/p in (3.3.201), we can show that (3.3.209) holds if $A_{k_j}(p) = \nabla_U^{\mathrm{End}(\mathbb{E}_p)} R_p/p$. We only have the case $A_{k_j}(p) = \nabla_U^{\mathrm{End}(\mathbb{E}_p)} D_p^2/p$ left to treat.

First, observe that for any operator A, it is equivalent to show that $\|As\|_{L^2} \leq C\|s\|_{L^2}$ for any section or for any section supported in a ball of radius $\varepsilon > 0$. We fix $x_0 \in X$, and $\varepsilon > 0$ as in Section 3.3.2, and we consider a section s supported in $B^X(x_0,\varepsilon)$. We will use here all the notations, identifications an trivializations of Section 3.3.2. We extend s by 0 to get an element of $\mathscr{C}^\infty_c(T_{\mathbb{R},x_0}X, \mathbb{E}_{x_0})$. To simplify, let us denote $\nabla_U^{\mathrm{End}(\mathbb{E}_p)} D_p^2/p(\lambda_0 - D_p^2/p)^{-1}$ by $A_p(\lambda_0)$. Let $\sigma_t = S_t^{-1}\kappa^{1/2}s$ and $\mathscr{A}_t(\lambda_0) = S_t^{-1}\kappa^{1/2}A_p(\lambda_0)\kappa^{-1/2}S_t$. We have

$$\|A_p(\lambda_0)s\|^2_{L^2} = t^{2n}\int_{\mathbb{R}^{2n}} \left|\kappa^{1/2}(A_p(\lambda_0)s)\right|^2 (tZ)dv_{TX}(Z) = t^{2n}\int_{\mathbb{R}^{2n}} \left|\mathscr{A}_t(\lambda_0)\sigma_t\right|^2 (Z)dv_{TX}(Z). \tag{3.3.210}$$

Thus, if we prove that

$$\|\mathscr{A}_t(\lambda_0)\|_t^{0,0} \leq C, \tag{3.3.211}$$

we will find

$$\|A_p(\lambda_0)s\|^2_{L^2} \leq Ct^{2n}\int_{\mathbb{R}^{2n}} \left|\sigma_t\right|^2 (Z)dv_{TX}(Z) = C\int_X |s|^2(x)dv_X(x) = C\|s\|^2_{L^2}, \tag{3.3.212}$$

which is the estimate we needed. To prove (3.3.211), observe that over $B^{T_{\mathbb{R},x_0}X}(0,\varepsilon)$ and under the identification $\mathbb{E}_p \simeq \mathbb{E}$, we have

$$\nabla^{\mathrm{End}(\mathbb{E}_p)} = \nabla^{\mathrm{End}(\mathbb{E})} = \nabla + [\Gamma_1, \cdot], \tag{3.3.213}$$

thus,

$$\nabla^{\mathrm{End}(\mathbb{E}_p)}(\nabla^p_{e_i}) = p(\nabla_U\Gamma^L)(e_i) + R^{\mathbb{E}}(U, e_i). \tag{3.3.214}$$

Hence, $\nabla^{\mathrm{End}(\mathbb{E}_p)}D_p^2/p$ has the form

$$\nabla^{\mathrm{End}(\mathbb{E}_p)}D_p^2/p = a_{i,j}(Z)\frac{1}{p}\nabla^{p,(0)}_{e_i}\nabla^{p,(0)}_{e_j} + \Big(\frac{1}{\sqrt{p}}b_j(Z) + \sqrt{p}c_j(Z)\Big)\frac{1}{\sqrt{p}}\nabla^{p,(0)}_{e_j} + \frac{1}{p}d(Z) + e(Z), \tag{3.3.215}$$

where $a_{i,j}$, b_j, c_j, d and e are bounded (along with their derivatives). Moreover, observe that $(\nabla_U\Gamma^L)(e_i)(Z) = O(|Z|)$ (apply [46, (1.2.30)] and observe that ∇_U only differentiate the parameter of the basis B), and that $c_j(Z)$ comes from the terms $(\nabla_U\Gamma^L)(e_i)$, so we have $c_j(0) = 0$. Using this fact and (3.3.215), we find that $t^{-1}c_j(tZ)$ is bounded as $t \to 0$ and that

$$\begin{aligned} S_t^{-1}\kappa^{1/2}\Big(\nabla^{\mathrm{End}(\mathbb{E}_p)}D_p^2/p\Big)\kappa^{-1/2}S_t = \\ a_{i,j}(tZ)\nabla^{(0)}_{t,e_i}\nabla^{(0)}_{t,e_j} + \big(b_j(tZ) + t^{-1}c_j(tZ)\big)\nabla^{(0)}_{t,e_j} + t^2d(tZ) + e(tZ), \end{aligned} \tag{3.3.216}$$

Using this structure, the fact that $\mathscr{A}_t(\lambda_0) = S_t^{-1}\kappa^{1/2}(\nabla^{\mathrm{End}(\mathbb{E}_p)}D_p^2/p)\kappa^{-1/2}S_t(\lambda_0 - \mathscr{L}_t^{(0)})^{-1}$ and arguments similar to those in the proof of Propositions 3.3.16-3.3.18, we find (3.3.211).

We have proved Lemma 3.3.29 for $\ell = 1$. The case $\ell \geq 1$ is similar. □

Proposition 3.3.30. *For any $\ell \in \mathbb{N}$, there exist $a, C > 0$ such that for $p \geq 1$ and $u \geq 1$,*

$$p^{-n}\big|\operatorname{Tr}_s[N_u\mathbb{K}_{p,u}]\big|_{\mathscr{C}^\ell(B)} \leq Ce^{-au}. \tag{3.3.217}$$

Proof. First, note that (3.3.200) is still true if we replace λ_0 by $\lambda \in \delta\cup\Delta$, and that the constant in the right hand side can be chosen independently of $\lambda_0 \in \Delta$, that is: there exists $C > 0$ such that

$$\left\|(\lambda - D_p^2/p)^{-1}\right\|_\infty \leq C, \quad \forall\lambda \in \delta\cup\Delta. \tag{3.3.218}$$

In the same way, (3.3.204) is also true if we replace λ_0 by $\lambda \in \Delta$ and we have $\sup_{\mu\geq\sqrt{c}}\frac{1+\mu}{|\lambda-\mu^2|} \leq C|\lambda|$, hence there exists $C > 0$ such that for $\lambda \in \delta\cup\Delta$,

$$\frac{1}{p}\left\|R_p(\lambda - D_p^2/p)^{-1}\right\|_\infty \leq C|\lambda|. \tag{3.3.219}$$

Thus by (3.3.205), (3.3.218) and (3.3.219), there exists $C > 0$ such that for $p \geq 1$ and $\lambda \in \delta \cup \Delta$,

$$\left\|(\lambda - C_p)^{-1}\right\|_\infty \leq C|\lambda|. \tag{3.3.220}$$

For $\lambda \in \Delta$ and $\lambda_0 \in \mathbb{R}_-^*$, we have

$$(\lambda - C_p)^{-1} = (\lambda_0 - C_p)^{-1} - (\lambda - \lambda_0)(\lambda_0 - C_p)^{-1}(\lambda - C_p)^{-1}. \tag{3.3.221}$$

In particular,

$$(\lambda - C_p)^{-q} = (\lambda_0 - C_p)^{-q}\left(1 - (\lambda - \lambda_0)(\lambda - C_p)^{-1}\right)^q. \tag{3.3.222}$$

From (3.3.191), (3.3.192) (3.3.220) and (3.3.222) we find that for $\lambda \in \delta \cup \Delta$,

$$\begin{aligned}\|(\lambda - C_p)^{-q}\|_1 &\leq \|(\lambda_0 - C_p)^{-q}\|_1 \left\|\left(1 - (\lambda - \lambda_0)(\lambda - C_p)^{-1}\right)^q\right\|_\infty \\ &\leq C|\lambda|^{2q}\left\|(\lambda_0 - C_p)^{-1}\right\|_q \\ &\leq C|\lambda|^{2q}p^n.\end{aligned} \tag{3.3.223}$$

On the other hand, we have

$$\mathbb{K}_{p,u} = \frac{1}{2i\pi}\psi_{1/\sqrt{u}}\int_\Delta \frac{(q-1)!}{(-u)^{q-1}}e^{-u\lambda}(\lambda - C_p)^{-q}d\lambda, \tag{3.3.224}$$

and there exist $\kappa, K > 0$ such that for $\lambda \in \delta \cup \Delta$,

$$\mathrm{Re}(\lambda) \geq K|\lambda| \geq \kappa. \tag{3.3.225}$$

From (3.3.223), (3.3.224) and (3.3.225) we deduce that there exist $a, C > 0$ such that for $p \geq 1$ and $u \geq 1$,

$$\begin{aligned}p^{-n}\left|\operatorname{Tr}_s\left[N_u\mathbb{K}_{p,u}\right]\right| &\leq p^{-n}C(1 + \sqrt{u}^{-n/2})\left\|\int_\Delta \frac{(q-1)!}{(-u)^{q-1}}e^{-u\lambda}(\lambda - C_p)^{-q}d\lambda\right\|_1 \\ &\leq p^{-n}C\int_\Delta |\lambda|^{2q}e^{-uK|\lambda|}\left\|(\lambda - C_p)^{-q}\right\|_1 d\lambda \\ &\leq Ce^{-au}.\end{aligned} \tag{3.3.226}$$

Proposition 3.3.30 is proved in the case where $\ell = 0$.

We now turn to the case $\ell = 1$. Equation (3.3.222) implies

$$\begin{aligned}\nabla_U^{\mathrm{End}(\mathbb{E}_p)}(\lambda - C_p)^{-q} = &\left[\nabla_U^{\mathrm{End}(\mathbb{E}_p)}(\lambda_0 - C_p)^{-q}\right]\left(1 - (\lambda - \lambda_0)(\lambda - C_p)^{-1}\right)^q \\ &+ (\lambda_0 - C_p)^{-q}\left[\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\left(1 - (\lambda - \lambda_0)(\lambda - C_p)^{-1}\right)^q\right].\end{aligned} \tag{3.3.227}$$

We claim that there is $C, N > 0$ such that for $\lambda \in \delta \cup \Delta$

$$\left\|\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\left(1 - (\lambda - \lambda_0)(\lambda - C_p)^{-1}\right)^q\right\|_\infty \leq C|\lambda|^N. \tag{3.3.228}$$

Indeed, the arguments of Propositions 3.3.16-3.3.18 that enables us to prove (3.3.211) from (3.3.216) also shows that (3.3.211) is still true if we replace therein λ_0 by $\lambda \in \delta \cup \Delta$ and that moreover there exists $N > 0$ such that $\|\mathscr{A}_t(\lambda)\|_t^{0,0} \leq C|\lambda|^N$. Hence, as in (3.3.212), we have $\|A_p(\lambda)\|_\infty \leq C|\lambda|^N$, i.e.,

$$\left\|\nabla_U^{\mathrm{End}(\mathbb{E}_p)}D_p^2/p(\lambda_0 - D_p^2/p)^{-1}\right\|_\infty \leq C|\lambda|^N. \tag{3.3.229}$$

Thus, decomposing $\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\left(1-(\lambda-\lambda_0)(\lambda-C_p)^{-1}\right)^q$ as a polynomial in λ whose coefficients have the form (3.3.207), and using (3.3.218), (3.3.219) and (3.3.229), we find (3.3.228).

Then, by (3.3.191), (3.3.192), (3.3.227) and (3.3.228), we find that there is $N'>0$ such that

$$p^{-n}\left\|\nabla^{\mathrm{End}(\mathbb{E}_p)}(\lambda-C_p)^{-q}\right\|_1 \leq C|\lambda|^{N'}. \tag{3.3.230}$$

Hence,

$$\begin{aligned} p^{-n}\left|\nabla^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}\operatorname{Tr}_s\left[N_u\mathbb{K}_{p,u}\right]\right| &= p^{-n}\left|\operatorname{Tr}_s\left[\nabla^{\mathrm{End}(\mathbb{E}_p)}(N_u\mathbb{K}_{p,u})\right]\right| \\ &\leq p^{-n}C\int_\Delta e^{-uK|\lambda|}\left\|\nabla^{\mathrm{End}(\mathbb{E}_p)}(\lambda-C_p)^{-q}\right\|_1 d\lambda \\ &\leq Ce^{-au}. \end{aligned} \tag{3.3.231}$$

This proves (3.3.217) for $\ell=1$.

The proof of Proposition 3.3.30 for $\ell\geq 1$ relies on similar arguments. □

The term involving $\mathbb{P}_{p,u}$

Proposition 3.3.31. *For any $\ell\in\mathbb{N}$, there is a $C>0$ such that for any $p\geq 1$ and $u\geq 1$,*

$$p^{-n}\left|\operatorname{Tr}_s\left[N_u\mathbb{P}_{p,u}\right]\right|_{\mathscr{C}^\ell(B)} \leq \frac{C}{\sqrt{u}}. \tag{3.3.232}$$

Proof. We first rewrite $\mathbb{P}_{p,u}$. As C_p has no eigenvalues between the two circles δ and δ/u, we have

$$\begin{aligned} \mathbb{P}_{p,u} &= \frac{1}{2i\pi}\psi_{1/\sqrt{u}}\int_{\delta/u} e^{-u\lambda}(\lambda-C_p)^{-1}d\lambda \\ &= \frac{1}{2i\pi}\psi_{1/\sqrt{u}}\int_{\delta} e^{-\lambda}(\lambda-uC_p)^{-1}d\lambda. \end{aligned} \tag{3.3.233}$$

We now use the technique of [8, Sect. 9.13]. Let $C_p^{(0)}=\frac{1}{p}D_p^2$ be the part of C_p of degree 0 in $\Lambda^\bullet(T^*_{\mathbb{R}}B)$. We denote by P_p the orthogonal projection form $\Omega^{0,\bullet}(X,\xi\otimes L^p)$ to the kernel of D_p^2, and $P_p^\perp=1-P_p$. We will make the abuse of notation $(C_p^{(0)})^{-1}=P_p^\perp(C_p^{(0)})^{-1}P_p^\perp$. Finally, we denote R_p/p by $\tilde{R}_p$. Then for $\lambda\in\delta$,

$$\begin{aligned} e^{-\lambda}(\lambda-uC_p)^{-1} &= \left(\sum_{k\geq 0}\frac{(-1)^k}{k!}\lambda^k\right)\left(\sum_{\ell\geq 0}(\lambda-uC_p^{(0)})^{-1}(u\tilde{R}_p)\dots(u\tilde{R}_p)(\lambda-uC_p^{(0)})^{-1}\right), \\ (\lambda-uC_p^{(0)})^{-1} &= \frac{1}{\lambda}P_p+(\lambda-uC_p^{(0)})^{-1}P_p^\perp. \end{aligned} \tag{3.3.234}$$

Moreover, $\lambda\mapsto(\lambda-uC_p^{(0)})^{-1}P_p^\perp$ is an holomorphic function on the interior of δ, so (3.3.234) yields to

$$\mathbb{P}_{p,u}=\psi_{1/\sqrt{u}}\sum_{\ell=0}^{\dim_{\mathbb{R}}B}\sum_{\substack{1\leq i_0\leq\ell+1\\ j_1,\dots,j_{\ell+1-i_0}\geq 0\\ \sum_{m=1}^{\ell+1-i_0}j_m\leq i_0-1}}\frac{(-1)^{\ell-\sum_m j_m}}{(i_0-1-\sum_m j_m)!}T_{p,1}(u\tilde{R}_p)T_{p,2}\dots(u\tilde{R}_p)T_{p,\ell+1}, \tag{3.3.235}$$

where P_p appears i_0 times among the $T_{p,j}$'s and the other terms are given respectively by $(uC_p^{(0)})^{-(1+j_1)}$,…,$(uC_p^{(0)})^{-(1+j_{\ell+1-i_0})}$.

As R_p is the part of positive degree of B_p^2 and $B_p^{(0)} = D_p$ (see (3.2.26)), we can decompose R_p with respect to the degree in $\Lambda^\bullet(T^*_{\mathbb{R}}B)$:

$$R_p = R_p^{(1)} + R_p^{(\geq 2)} \quad \text{with} \quad R_p^{(1)} = \left[B_p^{(1)}, D_p\right]. \tag{3.3.236}$$

We can rewrite the sum (3.3.235) as a sum of products of terms

$$\begin{aligned} &A_1(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(1)})A_2 \quad \text{or} \quad A_1(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})A_2, \\ &A_i \in \{P_p, (uC_p^{(0)})^{-(1+j)}, (uC_p^{(0)})^{-(1+j)/2}\}. \end{aligned} \tag{3.3.237}$$

Moreover, observe that

$$P_p\left[B_p^{(1)}, D_p\right]P_p = 0. \tag{3.3.238}$$

As a consequence, the possible degrees in u of a term $A_1(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(1)})A_2 = A_1(\sqrt{u}\tilde{R}_p^{(1)})A_2$ are:

$$\begin{cases} \deg_u P_p\sqrt{u}\tilde{R}_p^{(1)}P_p = -\infty, \\ \deg_u P_p\sqrt{u}\tilde{R}_p^{(1)}(uC_p^{(0)})^{-r} = \deg_u(uC_p^{(0)})^{-r}\sqrt{u}\tilde{R}_p^{(1)}P_p = \dfrac{1}{2} - r, \\ \deg_u(uC_p^{(0)})^{-r}\sqrt{u}\tilde{R}_p^{(1)}(uC_p^{(0)})^{-r'} = \dfrac{1}{2} - r - r'. \end{cases} \tag{3.3.239}$$

In any case, by (3.3.237), these terms are polynomials in $1/\sqrt{u}$.

Concerning the terms $A_1(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})A_2$, as $R_p^{(\geq 2)}$ is a sum of terms of degree greater than 2 in $\Lambda^\bullet(T^*_{\mathbb{R}}B)$ we find that the powers of u appearing are:

$$\begin{cases} u^{-j/2} & \text{in} \quad P_p(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})P_p, \\ u^{-r-j/2} & \text{in} \quad P_p(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})(uC_p^{(0)})^{-r} \text{ or } (uC_p^{(0)})^{-r}(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})P_p, \\ u^{-r-r'-j/2} & \text{in} \quad (uC_p^{(0)})^{-r}(u\psi_{1/\sqrt{u}}\tilde{R}_p^{(\geq 2)})(uC_p^{(0)})^{-r'}, \end{cases} \tag{3.3.240}$$

where $r, r' \in \frac{1}{2}\mathbb{N}^*$ and $2 \leq j \leq \dim_{\mathbb{R}} B$. This shows that $\mathbb{P}_{p,u}$ is in $\mathbb{C}_N\left[\frac{1}{\sqrt{u}}\right]$ for some uniform $N \in \mathbb{N}$. Furthermore, in each term of the sum (3.3.235) $i_0 \geq 1$ so P_p –which is a projector on a finite dimensional space– appears at least one time. Hence there exist $c_k(p) \in \Omega^\bullet(B)$ such that

$$p^{-n}\operatorname{Tr}_s\left[N_u\mathbb{P}_{p,u}\right] = \sum_{k=0}^{K} c_k(p)u^{-k/2}. \tag{3.3.241}$$

Moreover, by (3.3.203), we have for $r, r' \geq \frac{1}{2}$

$$\begin{cases} ||P_p\tilde{R}_pP_p||_\infty \leq C, \\ ||P_p\tilde{R}_p(C_p^{(0)})^{-r}||_\infty\,, \ ||(C_p^{(0)})^{-r}\tilde{R}_pP_p||_\infty \leq C\displaystyle\sup_{\mu\geq\sqrt{\nu}}\left((1+\mu)\mu^{-2r}\right) \leq C', \\ ||(C_p^{(0)})^{-r}\tilde{R}_p(C_p^{(0)})^{-r'}||_\infty \leq C''. \end{cases} \tag{3.3.242}$$

Therefore, each term in the sum (3.3.235) is a product of uniformly bounded terms, in which P_p appears at least once (because $i_0 \geq 1$). Thus,

$$|c_k(p)| \leq p^{-n}C\dim\ker(D_p^2) = p^{-n}C\dim H^0(X, \xi\otimes L^p) \leq C. \tag{3.3.243}$$

For the last inequality we have used Riemann-Roch-Hirzebruch theorem (see e.g. [46, Thm. 1.4.6]) and Kodaira vanishing theorem.

Finally, using Theorem 3.2.15, (3.3.189) and Proposition 3.3.30 we have for p large fixed

$$p^{-n}\operatorname{Tr}_s[N_u\mathbb{P}_{p,u}]\xrightarrow[u\to+\infty]{} p^{-n}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[N_V\exp(-(\nabla^{H(X,\xi\otimes L^p|_X)})^2)\right]=0. \tag{3.3.244}$$

Thus

$$c_0(p)=0. \tag{3.3.245}$$

Using (3.3.241), (3.3.243) and (3.3.245), we find (3.3.232) in the case $\ell=0$.

We now turn to the case $\ell=1$. By decomposing as above $\mathbb{P}_{p,u}$ in a sum of product of polynomial in $1/\sqrt{u}$, and then differentiating in the direction $U\in T_{\mathbb{R}}B$, we find that $\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\mathbb{P}_{p,u}$ is also a sum of product of polynomial in $1/\sqrt{u}$. Thus, here again there exist $c'_k(p)\in\Omega^\bullet(B)$ such that

$$p^{-n}\nabla_U^{\Lambda^\bullet(T_{\mathbb{R}}^*B)}\operatorname{Tr}_s[N_u\mathbb{P}_{p,u}]=p^{-n}\operatorname{Tr}_s\left[\nabla_U^{\mathrm{End}(\mathbb{E}_p)}N_u\mathbb{P}_{p,u}\right]=\sum_{k=0}^{K}c'_k(p)u^{-k/2}. \tag{3.3.246}$$

To conclude the proof as above, we need not only the uniform bounds given in (3.3.242), but also of the derivative of the terms appearing therein.

Firstly, , $\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\tilde{R}_p$ has the same structure as $\tilde{R}_p$ in (3.3.201), so all the estimates in (3.3.242) still holds if we replace therein R_p/p by $\nabla_U^{\mathrm{End}(\mathbb{E}_p)}\tilde{R}_p$.

Secondly, from (3.3.187), we know that

$$\begin{aligned}\nabla^{\mathrm{End}(\mathbb{E}_p)}P_p&=\frac{1}{2i\pi}\int_\delta\nabla^{\mathrm{End}(\mathbb{E}_p)}(\lambda-C_p^{(0)})^{-1}d\lambda\\&=-\frac{1}{2i\pi}\int_\delta(\lambda-C_p^{(0)})^{-1}\nabla^{\mathrm{End}(\mathbb{E}_p)}C_p^{(0)}(\lambda-C_p^{(0)})^{-1}d\lambda\end{aligned} \tag{3.3.247}$$

Using (3.3.218), (3.3.229) and (3.3.247), we find

$$\|\nabla^{\mathrm{End}(\mathbb{E}_p)}P_p\|_\infty\leq C. \tag{3.3.248}$$

Finally, we study $\nabla^{\mathrm{End}(\mathbb{E}_p)}(C_p^{(0)})^{-r}$. Observe that in decomposition in (3.3.237) and (**??**), we introduce half-integer exponent to treat the different terms more concisely, but in fact we can only consider the case $r\in\mathbb{N}^*$. Thus, we can prove in the same way as we proved (3.3.212) that $\|\nabla^{\mathrm{End}(\mathbb{E}_p)}(C_p^{(0)})^{-1}\|_\infty\leq C$, and with (3.3.187) we now that $\|(C_p^{(0)})^{-1}\|_\infty\leq C$. In conclusion, we have

$$\|\nabla^{\mathrm{End}(\mathbb{E}_p)}(C_p^{(0)})^{-r}\|_\infty\leq C. \tag{3.3.249}$$

With (3.3.242), (3.3.248) and (3.3.249), we can conclude the proof of Proposition 3.3.31 when $\ell=1$. For $\ell\geq1$, the reasoning is similar. □

With (3.3.189) and Propositions 3.3.30 and 3.3.31, we have proved Theorem 3.3.25.

3.4 Torsion forms associated with a direct image.

The purpose of this section is to prove Theorem 3.1.7.

We recall some notations. Let N, M and B be three complex manifolds. Let $\pi_1\colon N\to M$ and $\pi_2\colon M\to B$ be holomorphic fibrations with compact fiber Y and X respectively. Then $\pi_3:=\pi_2\circ\pi_1\colon N\to B$ is a holomorphic fibration, whose compact fiber is denoted by

Z. We denote by n_X (resp. n_Y, n_Z) the complex dimension of X (resp. Y, Z). Note that $\pi_1|_Z\colon Z \to X$ is a holomorphic fibration with fiber Y. This is summarized in the following diagram:

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"] & N \arrow[d, "\pi_1"] \arrow[dr, "\pi_3"] & \\ & X \arrow[r] & M \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Let (π_2, ω^M) be a structure of Hermitian fibration (see Section 3.2.1). We denote by $T_B^H M$ the corresponding horizontal space.

Let (ξ, h^ξ) be a holomorphic Hermitian vector bundle on M, and let (η, h^η) be a holomorphic Hermitian vector bundle on N. Let (L, h^L) be a holomorphic Hermitian line bundle on N. We denotes its Chern connection by ∇^L, and the corresponding curvature by R^L. By Assumption 3.1.4, L is positive along the fibers of π_3. In particular, $\frac{\sqrt{-1}}{2\pi} R^L$ defines metric $g^{T_\mathbb{R} Z}$ on $T_\mathbb{R} Z$, by the formula

$$g^{T_\mathbb{R} Z}(U,V) = \frac{\sqrt{-1}}{2\pi} R^L(U, J^{T_\mathbb{R} Z} V)\,, \quad U, V \in T_\mathbb{R} Z. \tag{3.4.1}$$

Similarly, we get a metric $g^{T_\mathbb{R} Y}$ on $T_\mathbb{R} Y$.

Recall that

$$T_B^H N = (TZ)^\perp, \quad T_M^H N = (TY)^\perp, \quad T_X^H Z = T_M^H N \cap TZ, \tag{3.4.2}$$

where the orthogonal complements are taken with respect to R^L. Also, $\dot{R}^{X,L} \in \pi_3^* \mathrm{End}(TX)$ is the Hermitian matrix such that for any $U, V \in TX$, if we denote their horizontal lifts by $U^H, V^H \in T_X^H Z$, then

$$R^L(U^H, \overline{V}^H) = \langle \dot{R}^{X,L} U, V \rangle_{h^{TX}}. \tag{3.4.3}$$

By Assumption 3.1.4, $\dot{R}^{X,L}$ is positive definite. Finally,

$$\Theta^N = \frac{\sqrt{-1}}{2\pi} R^L \quad \text{and} \quad \Theta^Z = \frac{\sqrt{-1}}{2\pi} R^L|_{T_\mathbb{R} Z \times T_\mathbb{R} Z}. \tag{3.4.4}$$

Recall that we have assumed that (for p large) the direct image both $R^\bullet \pi_{1,*}(\eta \otimes L^p)$ is locally free. Let $F_p := H^\bullet(Y, (\eta \otimes L^p)|_Y)$ the corresponding bundle, endowed with the L^2 metric h^{F_p} induced by h^η, h^L and $g^{T_\mathbb{R} Y}$.

We have also assumed that (for p large) $R^\bullet \pi_{2,*}(\xi \otimes F_p)$ of is locally free and that we have $R^\bullet \pi_{2,*}(\xi \otimes F_p) \simeq R^\bullet \pi_{3,*}(\pi_1^* \xi \otimes \eta \otimes L^p)$.

The objects corresponding to this situation will be denoted by

$$\begin{aligned}
&E_{p,b}^k = \mathscr{C}^\infty\left(X_b, \left(\Lambda^{0,k}(T^*X) \otimes \xi \otimes F_p\right)|_{X_b}\right),\\
&\nabla^p = \nabla^{E_p, LC},\\
&\bar\partial^p = \text{Dolbeault operator of } E_p,\\
&D_p = \bar\partial^p + \bar\partial^{p,*},\\
&B_p\,,\ B_{p,u} = \text{associated superconnections as in (3.2.24)},\\
&\nabla_u^p = \text{connection corresponding to (3.2.33) associated with } \xi \otimes F_p = \nabla_u \otimes 1 + 1 \otimes \nabla^{F_p}.
\end{aligned} \tag{3.4.5}$$

Then we can construct as in Section 3.2 the holomorphic analytic torsion forms $\mathscr{T}(\omega^M, h^{\xi \otimes F_p})$ associated with ω^M and $(\xi \otimes F_p, h^{\xi \otimes F_p})$.

Theorem 3.1.7 is the family version of [18], with a more general bundle. Indeed, let V is a positive bundle on X in the sense of [18]. Then on the projectivization $N := \mathbb{P}(V^*)$ of V^* we can define L to be the dual of the universal line bundle. Then L satisfies Assumption 3.1.4. Let Y be the fiber of $\mathbb{P}(V^*) \to M$, then for any $p \in \mathbb{N}$, $H^\bullet(Y, L^p|_Y) = H^0(Y, L^p|_Y) \simeq S^p(V)$ the p^{th} symmetric power of V. Note that in [18], they used a trick due to Getzler [33] to transfer the problem from $(M, S^p(V))$ to $(\mathbb{P}(V^*), L^p)$ and then use the results of [17]. In fact, it is easy to see that the results of [18] apply to the direct image of powers of a line bundle on a bigger manifold given by a principal G-bundle with G compact and connected. Let us now explain why we cannot use the same approach here.

In [18], they consider a G-principal bundle $P \to X$, with G compact and connected, and a holomorphic fibration $Z = P \times_G Y \to X$ given by an unitary and holomorphic action of G on a manifold Y. They also consider a positive line bundle $(L', h^{L'})$ on Y, such that the G-action on Y lift to an isometric G-action on L'. Then one can construct the line bundle $L := P \times_G L'$ on N. Bismut and Vasserot then study the torsion associated with $F_p = H^0(Y, L^p)$ (for simplicity we omit the twisting bundle ξ here). Their method consist in decomposing the Laplacian $\square_p^Z$ acting on $\Omega^{0,\bullet}(Z, L^p)$ as the sum of two commuting operators:

$$\square_p^Z = \square_p^Y + \square^{\Omega^{0,\bullet}(Y,L^p)}, \tag{3.4.6}$$

where $\square_p^Y$ is the Laplacian along the fiber, and $\square^{\Omega^{0,\bullet}(Y,L^p)}$ is a Laplacian-like operator acting on the infinite-dimensional bundle $\Omega^{0,\bullet}(Y, L^p)$, this two operators being extended to act on $\mathscr{C}^\infty(Y, \pi^*\Lambda^{0,\bullet}(T^*X) \otimes \Lambda^{0,\bullet}(T^*Y), L^p) \simeq \Omega^{0,\bullet}(Z, L^p)$. The key points are first that, as this operators commute, $\square^{\Omega^{0,\bullet}(Y,L^p)}$ acts on each eigenspace of $\square_p^Y$, and second that the restriction of $\square^{\Omega^{0,\bullet}(Y,L^p)}$ to $\ker(\square_p^Y) \simeq H^0(Y, L^p) \otimes \pi^*\Lambda^{0,\bullet}(T^*X)$ is precisely $\square_p^X$, the Laplacian acting on $\Omega^{0,\bullet}(X, F_p)$. Using these facts and decomposing the trace of $\exp(-u\square_p^Z)$ as a sum on each eigenspace of $\square_p^Y$, they find

$$\operatorname{Tr}\left[\exp(-u\square_p^X|_{\Omega^{0,k}(X,F_p)})\right] = \sum_{\ell=0}^{\dim Y} (-1)^\ell \operatorname{Tr}\left[\exp(-u\square_p^Z|_{\Omega^{0,k}(X,\Omega^{0,\ell}(Y,L^p))})\right]. \tag{3.4.7}$$

Thus, the computation of the asymptotic of the torsion on X associated with F_p can be deduced from the asymptotic of the torsion on Z associated with L^p.

To get (3.4.6), the main ingredient is that G acts by holomorphic isometry, and thus we need the hypothesis of G being compact. Moreover, even to give the complex structure on Z, one has to use the complexification $G_\mathbb{C}$ of G, which exist if G is compact and connected (see [37, Thm. 4.1]). Indeed we can extend in this case the actions of G to actions of $G_\mathbb{C}$ and show that their is a $G_\mathbb{C}$-bundle $P_\mathbb{C}$ such that P is a reduction of $P_\mathbb{C}$, and then $P \times_G Y = P_\mathbb{C} \times_{G_\mathbb{C}} Y$.

Here we do not make any assumption on G, and as a consequence, we cannot use the same trick to reduce the problem to our first result. But furthermore, in the family setting, even if N is constructed via a G-principal bundle on M with G compact and connected, we cannot use the same method. Indeed, the “differential form along the basis” part of the operators introduce a nilpotent part acting along the fiber, and we cannot decompose the traces as above as a sum on each eigenspace of the operator along the fiber. Let us show this on an example. Take the two fibrations to be $N = P \times_G Y \to M$ and $M = B$. Let $L \to N$ be a line bundle which is positive along the fiber $Z = Y$. Take $\omega^M = 0$. Then we denote by B_p^N the Bismut superconnection associated with Θ^N and L^p, and B_p^M the Bismut superconnection associated with ω^M and the infinite-dimentional bundle

$\Omega^{0,\bullet}(Y, L^p)$. Then the analogue of (3.4.6) reads

$$B_p^N = B_p^M + \square_p^Y - \frac{c(T)}{2\sqrt{2}}, \tag{3.4.8}$$

where T is defined for the fibration $N \to M$ as in (3.2.13). Here, B_p^M indeed commutes with $\square_p^Y - \frac{c(T)}{2\sqrt{2}}$ (see [2, Sect. 10.7]), but the operator acting along the fiber $\square_p^Y - \frac{c(T)}{2\sqrt{2}}$ is no longer diagonalisable.

The strategy of proof of Theorem 3.1.7 will be formally the same as for Theorem 3.1.3. However, the main difficulty is that in the case of a line bundle (that is $Y = \{*\}$), $F_p = L^p$ is of constant dimension 1 so locally the operators have their coefficients in a fixed space (see Remark 3.3.6), whereas it is not the case here. To overcome this issue, we will use an approach inspired by [15, 16], that is we will consider all the operators depending on p at once with the formalism of Toeplitz operators of [46]. More precisely, we will consider the family $\{B^2_{p,u}, p \in \mathbb{N}\}$ as a differential operators with coefficient in the Toeplitz algebra (see (3.4.52)). A crucial point is to use the operator norm on matrices to have boundedness properties of Toeplitz operators. Here, the first difficulty is that there is no longer a limiting operator (as the space changes), but we can show that instead there is an asymptotic operator with Toepltiz coefficients. The problem is then that we cannot compute its heat kernel explicitly (with comparison to (3.3.125)), but using the properties of operator with Toeplitz coefficients developed in Section 3.4.4, we can nonetheless give an asymptotic formula. An other difficulty comes from the fact that we cannot use the same method to prove the uniform development of the heat kernel as $u \to 0$ as we did before (see the proofs of Theorems 3.3.23 and 3.4.24), and we cannot hope to prove that the coefficients converges. Instead, we prove that the coefficients are asymptotic to certain Toepltiz operators.

Once again, to simplify the statements in the following, we will assume that B is compact. However, the reader should be aware of the fact that the constants appearing in the sequel depends on the compact subset of B we are working on.

This section is organized as follows. In Subsection 3.4.1, we show how to use Theorem 3.1.3, [13] and [41] to compute the limit modulo ∂ and $\bar{\partial}$ exact forms of the torsion forms, in the case where B is compact Kähler and where (π_2, ω^M) is a Kähler fibration. In Subsections 3.4.2 and 3.4.3, we recall the formalism of Toepltiz operators. In Subsection 3.4.4, we introduce operators with Toeplitz coefficients and show some properties of their Schwartz kernels. In Subsection 3.4.5, we show that our problem is local. In Subsection 3.4.6, we rescale the Bismut superconnection and compute the limit operator, then we obtain the convergence of the heat kernel in Theorem 3.1.9. Then, in Subsection 3.4.7, we prove our main theorem, using two results which are proved in Subsections 3.4.8 and 3.4.9.

3.4.1 The case where the basis is compact Kähler and the fibration is Kähler

We keep here all the notations and hypothesis above, but we assume in addition that ω^M is closed, so that (π_2, ω^M) is a Kähler fibration. Set $\mu = \pi_1^*\xi \otimes \eta$, and let h^μ be the metric on μ induced by h^ξ and h^η.

Recall that Q^B and $Q^{B,0}$ are defined in Definition 3.2.12. In Subsection 3.4.1, we derive in this section Theorem 3.1.7 in $Q^B/Q^{B,0}$. The idea is to use [13] and [41] to express $\mathscr{T}(\omega^M, h^{\xi\otimes F_p})$ in terms of torsions associated with $\mu \otimes L^p$, then apply Theorem

3.1.3 to get the asymptotic. It is important to keep in mind that this method cannot prove the convergence at the form level of the torsion forms, and that when B is not compact or not Kähler , the space $Q^{B,0}$ is not closed, and a limit in $Q^B/Q^{B,0}$ is not relevant. We thus assume that B is compact and Kähler .

Note however that in degree zero, i.e., for the torsion of Ray-Singer, we do not have this problem of taking quotient, so the theorem of [41] in degree 0 (which was first proved in [3]) gives Theorem 3.1.7 in degree 0.

Let us first recall some facts about characteristic classes. Let S be a complex manifold and let $E \to S$ be a holomorphic vector bundle. We endow E with a metric h, and we denote the corresponding Chern connection by ∇^E and its curvature by R^E. The first Chern form $c_1(E,h)$, the Chern character $\mathrm{ch}(E,h)$ and the Todd genus $\mathrm{Td}(E,h)$ of (E,h) are the differential forms defined respectively by

$$\begin{aligned} c_1(E,h) &= -\operatorname{Tr}\left[\frac{R^E}{2i\pi}\right], \\ \mathrm{ch}(E,h) &= \operatorname{Tr}\left[\exp\left(-\frac{R^E}{2i\pi}\right)\right], \\ \mathrm{Td}(E,h) &= \det\left(\frac{R^E/2i\pi}{\exp(R^E/2i\pi)-1}\right). \end{aligned} \tag{3.4.9}$$

Then we have

$$\begin{aligned} \mathrm{ch}(E,h) &= \mathrm{rk}(E) + c_1(E,h) + (\deg \geq 4), \\ \mathrm{Td}(E,h) &= 1 + \frac{1}{2}c_1(E,h) + (\deg \geq 4). \end{aligned} \tag{3.4.10}$$

We endow E with two metrics h_1 and h_2. Let $A \in \mathrm{End}(E)$ be such that $h_1(\cdot\,,\cdot) = h_2(A\cdot\,,\cdot)$. We denote by ∇_i^E the corresponding Chern connection and by R_i^E their curvatures ($i=1,2$). In [10], for any polynomial $f \in \mathbb{C}[T]$, they define a universal Bott-Chern classes $\widetilde{f}(E,h_1,h_2)$ such that

$$\frac{\bar{\partial}\partial}{2i\pi}\widetilde{f}(E,h_1,h_2) = \mathrm{Tr}[f(R_1^E/2i\pi)] - \mathrm{Tr}[f(R_2^E/2i\pi)]. \tag{3.4.11}$$

Observe that if $h_i^{\det E}$ is the metric on $\det E$ induced by h_i, then $c_1(E,h_i) = c_1(\det E, h_i^{\det E})$. In particular, by universality and (3.4.10), we have

$$\begin{aligned} \widetilde{\mathrm{ch}}(E,h_1,h_2)^{(0)} &= \widetilde{c_1}(\det E, h_1^{\det E}, h_2^{\det E}), \\ \widetilde{\mathrm{Td}}(E,h_1,h_2)^{(0)} &= \frac{1}{2}\widetilde{c_1}(\det E, h_1^{\det E}, h_2^{\det E}). \end{aligned} \tag{3.4.12}$$

Let R_i be the Chern curvature of $h_i^{\det E}$. As $h_2^{\det E} = \det A \times h_1^{\det E}$, we have $R_2 = R_1 + \partial\bar{\partial}\log\det A$, and thus

$$\widetilde{\mathrm{ch}}(E,h_1,h_2)^{(0)} = \log\det A = 2\widetilde{\mathrm{Td}}(E,h_1,h_2)^{(0)}. \tag{3.4.13}$$

Let us now turn back to our problem.

With our three fibrations $\pi_1 \colon N \to M$, $\pi_2 \colon M \to B$ and $\pi_3 = \pi_2 \circ \pi_1 \colon N \to B$, and the 2-forms ω^M on M and Θ^N on N, we are in the situation of [41], and more precisely in the case i) of p.542 if $p \geq p_0$.

Let g^{TY} and g^{TZ} are the metric on TY and TZ induced by Θ^N. Let g^{TX} be the metric of TX induced by ω^M. Let $\widetilde{\mathrm{Td}}(TZ, TX, g^{TZ}, g^{TX}) \in Q^N/Q^{N,0}$ and be the Bott-Chern class associated with the metrics g^{TZ} and $\pi_1^* g^{TX} \oplus g^{TY}$ on TZ, that is

$$\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}(TZ, TX, g^{TZ}, g^{TX}) = \mathrm{Td}(TZ, g^{TZ}) - \pi_1^*(\,\mathrm{Td}(TX, g^{TX}))\,\mathrm{Td}(TY, g^{TY}). \quad (3.4.14)$$

Let $h_p^{H^0} = h_{h^\mu}^{H^0(Z,\eta\otimes L^p)}$ be the metric on $H^0(Z, \mu\otimes L^p)$ induced by h^L, h^μ and $g^{T_{\mathbb{R}}Z}$, and let $\tilde{h}_p^{H^0}$ be the metric on $H^0(X, H^0(Y, \mu\otimes L^p))$ induced by h^L, h^μ, $g^{T_{\mathbb{R}}X}$ and $g^{T_{\mathbb{R}}Y}$. Let $\varphi\colon Z \to \mathbb{R}$ be such that $dv_X dv_Y = e^{-\varphi} dv_Z$, that is

$$\varphi = \log\left[\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\right]. \quad (3.4.15)$$

Set $\tilde{h}^\mu = e^{-\varphi} h^\mu$. Under the isomorphism $H^0(Z, \mu\otimes L^p) \simeq H^0(X, H^0(Y, \mu\otimes L^p))$, we then have

$$\tilde{h}_p^{H^0} = h_{\tilde{h}^\mu}^{H^0(Z,\eta\otimes L^p)}. \quad (3.4.16)$$

We denote by $\widetilde{\mathrm{ch}}(H^0(Z, \mu\otimes L^p), \tilde{h}_p^H, h_p^H) \in Q^B/Q^{B,0}$ the Bott-Chern class such that

$$\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}(H^0(Z, \mu\otimes L^p), \tilde{h}_p^{H^0}, h_p^{H^0}) = \mathrm{ch}(H^0(Z, \mu\otimes L^p), \tilde{h}_p^{H^0}) - \mathrm{ch}(H^0(Z, \mu\otimes L^p), h_p^{H^0}). \quad (3.4.17)$$

Let $\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p})$ and $\mathscr{T}_B(\Theta^N, h^{\mu\otimes L^p})$ be the torsion forms associated with $(\pi_1, \Theta^N, h^{\mu\otimes L^p})$ and $(\pi_3, \Theta^N, h^{\mu\otimes L^p})$. Then [41, Thm. 0.1] states that in $Q^B/Q^{B,0}$,

$$\begin{aligned}\mathscr{T}(\omega^M, h^{\xi\otimes F_p}) &= \mathscr{T}_B(\Theta^N, h^{\mu\otimes L^p}) - \int_X \mathrm{Td}(TX, g^{TX})\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p}) \\ &+ \int_Z \widetilde{\mathrm{Td}}(TZ, TX, g^{TZ}, g^{TX})\,\mathrm{ch}(\mu\otimes L^p, h^{\mu\otimes L^p}) - \widetilde{\mathrm{ch}}(H^0(Z, \mu\otimes L^p), \tilde{h}_p^{H^0}, h_p^{H^0}). \quad (3.4.18)\end{aligned}$$

Also, by (3.4.16) and [13, Thm 3.10], we have in $Q^B/Q^{B,0}$

$$\begin{aligned}\widetilde{\mathrm{ch}}(H^0(Z, \mu\otimes L^p), \tilde{h}_p^{H^0}, h_p^{H^0}) &= \mathscr{T}_B(\Theta^N, h^{\mu\otimes L^p}) - \mathscr{T}_B(\Theta^N, \tilde{h}^{\mu\otimes L^p}) \\ &- \int_Z \mathrm{Td}(TZ, g^{TZ})\widetilde{\mathrm{ch}}(\mu\otimes L^p, \tilde{h}^{\mu\otimes L^p}, h^{\mu\otimes L^p}). \quad (3.4.19)\end{aligned}$$

Thus, (3.4.18) and (3.4.19) shows that in $Q^B/Q^{B,0}$ we have

$$\begin{aligned}\mathscr{T}(\omega^M, h^{\xi\otimes F_p}) &= \mathscr{T}_B(\Theta^N, \tilde{h}^{\mu\otimes L^p}) - \int_X \mathrm{Td}(TX, g^{TX})\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p}) \\ &+ \int_Z \widetilde{\mathrm{Td}}(TZ, TX, g^{TZ}, g^{TX})\,\mathrm{ch}(\mu\otimes L^p, h^{\mu\otimes L^p}) \\ &+ \int_Z \mathrm{Td}(TZ, g^{TZ})\widetilde{\mathrm{ch}}(\mu\otimes L^p, \tilde{h}^{\mu\otimes L^p}, h^{\mu\otimes L^p}). \quad (3.4.20)\end{aligned}$$

We now compute the asymptotic of each term in the right hand side of (3.4.20). Consider the following orthogonal decomposition with respect to Θ^N:

$$\begin{aligned} TN &= TZ \oplus T_B^H N, & TZ &= TY \oplus T_X^H Z, \\ TN &= TY \oplus T_M^H N, & T_M^H N &= T_B^H N \oplus T_X^H Z, \end{aligned} \quad (3.4.21)$$

and set

$$\begin{aligned}\Theta^X &= \Theta^N|_{T^H_X Z\times T^H_X Z}, \quad \Theta^Y = \Theta^N|_{T_\mathbb{R} Y\times T_\mathbb{R} Y},\\ \Theta^M &= \Theta^N|_{T^H_M N\times T^H_M N}.\end{aligned} \tag{3.4.22}$$

Using the decompositions (3.4.21), we can extend these forms (by 0) to $TN\times TN$. Then,

$$\Theta^N = \Theta^Y + \Theta^M = \Theta^Y + \Theta^X + \Theta^B = \Theta^Z + \Theta^B. \tag{3.4.23}$$

Using Theorem 3.1.3 we have

$$\mathscr{T}_B(\Theta^N, \tilde{h}^{\mu\otimes L^p})^{(2k)} = n_Z p^{n_Z+k}\log p \frac{\mathrm{rk}(\mu)}{2}\left(\int_Z e^{\Theta^B}\frac{\Theta^{Z,n_Z}}{n_Z!}\right)^{(2k)} + o(p^{n_Z+k}). \tag{3.4.24}$$

In deed the matrix $\dot{R}^{Z,L}/2\pi$ which should appear in Theorem 3.1.3 is the identity as the metric in Z is already given by the curvature R^L.

For $\alpha\in\Lambda^\bullet(T^*M)\simeq\Lambda^\bullet((T^HB)^*)\otimes\Lambda^\bullet(T^*X)$, we denote by $\alpha^{(i,j)}$ the projection of α in $\Lambda^i((T^HB)^*)\otimes\Lambda^j(T^*X)$. Then we have by Theorem 3.1.3

$$\begin{aligned}&\left(\int_X \mathrm{Td}(TX,g^{TX})\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p})\right)^{(2k)}\\ &= \sum_{j,\ell}\int_X \mathrm{Td}(TX,g^{TX})^{(2\ell,2j)}\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p})^{(2(k-\ell),2(n_X-j))}\\ &= \sum_{j,\ell} n_Y p^{n_Y+k-\ell+n_X-j}\log p\frac{\mathrm{rk}(\mu)}{2}\int_X \mathrm{Td}(TX,g^{TX})^{(2\ell,2j)}\left(\int_Y e^{\Theta^M}\frac{\Theta^{Y,n_Y}}{n_Y!}\right)^{(2(k-\ell),2(n_X-j))}\\ &\qquad + o(p^{n_Y+k-\ell+n_X-j}).\end{aligned} \tag{3.4.25}$$

Thus, the leading term is the term with $j=\ell=0$. Using $\mathrm{Td}^{(0)}=1$ and (3.4.23), we find that

$$\begin{aligned}&\left(\int_X \mathrm{Td}(TX,g^{TX})\mathscr{T}_M(\Theta^N, h^{\mu\otimes L^p})\right)^{(2k)}\\ &\qquad = n_Y p^{n_Z+k}\log p\frac{\mathrm{rk}(\mu)}{2}\left(\int_X\int_Y e^{\Theta^B}\frac{\Theta^{X,n_X}}{n_X!}\frac{\Theta^{Y,n_Y}}{n_Y!}\right)^{(2k)} + o(p^{n_Z+k})\\ &\qquad\qquad = n_Y p^{n_Z+k}\log p\frac{\mathrm{rk}(\mu)}{2}\left(\int_Z e^{\Theta^B}\frac{\Theta^{Z,n_Z}}{n_Z!}\right)^{(2k)} + o(p^{n_Z+k}).\end{aligned} \tag{3.4.26}$$

From (3.4.13) and (3.4.15), we have

$$\begin{aligned}\widetilde{\mathrm{ch}}(\mu,\tilde{h}^\mu,h^\mu)^{(0)} &= \mathrm{rk}(\mu)\varphi = \mathrm{rk}(\mu)\log\left[\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\right],\\ \widetilde{\mathrm{Td}}(TZ,TX,g^{TZ},g^{TX})^{(0)} &= -\frac{1}{2}\log\left[\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\right].\end{aligned} \tag{3.4.27}$$

Now, using a similar argument that in (3.4.25) and (3.4.27), we find

$$\begin{aligned}\Big(\int_Z \widetilde{\mathrm{Td}}(TZ,TX,g^{TZ},g^{TX})\,\mathrm{ch}(\mu\otimes L^p,h^{\mu\otimes L^p})\Big)^{(2k)}\\ &= \Big(\int_Z \widetilde{\mathrm{Td}}(TZ,TX,g^{TZ},g^{TX})\,\mathrm{ch}(\mu,h^{\mu})e^{p\Theta^N}\Big)^{(2k)}\\ &= p^{n_Z+k}\int_Z \widetilde{\mathrm{Td}}(TZ,TX,g^{TZ},g^{TX})^{(0)}\,\mathrm{ch}(\mu,h^{\mu})^{(0)}\frac{\Theta^{Z,n_Z}}{n_Z!}\frac{\Theta^{B,k}}{k!}+o(p^{n_Z+k})\\ &= -p^{n_Z+k}\frac{\mathrm{rk}(\mu)}{2}\Big(\int_Z\log\Big[\det\Big(\frac{\dot{R}^{X,L}}{2\pi}\Big)\Big]\frac{\Theta^{Z,n_Z}}{n_Z!}e^{\Theta^B}\Big)^{(2k)}+o(p^{n_Z+k}).\end{aligned} \tag{3.4.28}$$

In the same way, using $\widetilde{\mathrm{ch}}(\mu\otimes L^p,\tilde{h}^{\mu\otimes L^p},h^{\mu\otimes L^p})=\widetilde{\mathrm{ch}}(\mu,\tilde{h}^{\mu},h^{\mu})\,\mathrm{ch}(L,h^L)^p$, we find

$$\begin{aligned}\Big(\int_Z \mathrm{Td}(TZ,g^{TZ})\widetilde{\mathrm{ch}}(\mu\otimes L^p,\tilde{h}^{\mu\otimes L^p},h^{\mu\otimes L^p})\Big)^{(2k)}\\ &= \Big(\int_Z \mathrm{Td}(TZ,g^{TZ})\widetilde{\mathrm{ch}}(\mu,\tilde{h}^{\mu},h^{\mu})e^{p\Theta^N}\Big)^{(2k)}\\ &= p^{n_Z+k}\int_Z \mathrm{Td}(TZ,g^{TZ})^{(0)}\widetilde{\mathrm{ch}}(\mu,\tilde{h}^{\mu},h^{\mu})^{(0)}\frac{\Theta^{Z,n_Z}}{n_Z!}\frac{\Theta^{B,k}}{k!}+o(p^{n_Z+k})\\ &= p^{n_Z+k}\,\mathrm{rk}(\mu)\Big(\int_Z\log\Big[\det\Big(\frac{\dot{R}^{X,L}}{2\pi}\Big)\Big]\frac{\Theta^{Z,n_Z}}{n_Z!}e^{\Theta^B}\Big)^{(2k)}+o(p^{n_Z+k}).\end{aligned} \tag{3.4.29}$$

Using (3.4.23), (3.4.24), (3.4.26), (3.4.28) and (3.4.29) we find Theorem 3.1.7 in $Q^B/Q^{B,0}$.

3.4.2 The algebra of Toeplitz operators

In this subsection, we describe the formalism of Toeplitz operator introduced by Berezin [1] and Boutet de Monvel-Guillemin [21], and developed by Bordemann-Meinrenken-Schlichenmaier [19], Schlichenmaier [55] and Ma-Marinescu [46], [48].

We fix $m\in M$ for this subsection, and we denote Y_m simply by Y.

Thus, we are given a complex manifold Y of dimension n_Y, endowed with an Hermitian vector bundle $(\eta,h^\eta)|_Y$ and with a positive line bundle $(L,h^L)|_Y$. Recall that R^L is the Chern curvature of L and that

$$\Theta^Y=\frac{\sqrt{-1}}{2\pi}R^L|_{T_{\mathbb{R}}Y\times T_{\mathbb{R}}Y}, \tag{3.4.30}$$

and $g^{T_{\mathbb{R}}Y}=\Theta^Y(\cdot,J\cdot)$ is the associated metric.

Let

$$\mathcal{A}=\mathscr{C}^\infty(Y,\mathrm{End}(\eta)), \tag{3.4.31}$$

which we endow $\mathcal{A}$ with the L^2-metric induced by $g^{T_{\mathbb{R}}Y}$, h^L and h^η.

For $p\in\mathbb{N}$ and $A\in\mathrm{End}(L^2(Y,\eta\otimes L^p))$, we will use the same notations as in Definition 3.3.27, i.e., $||A||_\infty$ denotes the operator norm of A and $||A||_1$ its trace norm (if A is trace class).

Let P_p be the orthogonal projection form $L^2(Y,\eta\otimes L^p)$ onto $H^0(Y,\eta\otimes L^p)$. By Riemann-Roch-Hirzebruch theorem and Kodaira vanishing theorem, we now that $\dim F_p\leq Cp^{n_Y}$, thus if $A\in\mathrm{End}(L^2(Y,\eta\otimes L^p))$ is such that $P_pAP_p=A$, we have

$$||A||_1\leq C||A||_\infty p^{n_Y}. \tag{3.4.32}$$

If (V, h^V) is any finite dimensional Hermitian vector space and if $u \in \mathrm{End}(V)$, we denote by $\|u\|$ the operator norm of u.

For $f \in \mathcal{A}$, set

$$\|f\|_{\mathscr{C}^0} = \sup_{y \in Y} \|f(y)\|. \tag{3.4.33}$$

This defines a metric on $\mathcal{A}$.

For $f \in \mathcal{A}$, we denote by $T_{f,p}$ the *Berezin-Toeplitz quantization* of f, that is

$$T_{f,p} = P_p f P_p. \tag{3.4.34}$$

Observe that

$$||T_{f,p}||_\infty \leq \|f\|_{\mathscr{C}^0}. \tag{3.4.35}$$

Moreover, by [46, (4.1.84), Lem. 7.2.4], as $p \to +\infty$, we have

$$\mathrm{Tr}^{F_p}[T_{f,p}] = p^{n_Y} \int_Y \mathrm{Tr}^\eta[f] e^{\Theta^Y} + O(p^{n_Y - 1}). \tag{3.4.36}$$

Recall that Toeplitz operators are defined in Definition 3.1.8. As in [46], for a Toeplitz operator T_p with corresponding sections f_r, we will use the notation

$$T_p = \sum_{r=0}^{+\infty} p^{-r} T_{f_r,p} + O(p^{-\infty}). \tag{3.4.37}$$

We denote by $\mathcal{T}$ the space of Toeplitz operators on Y.

It follows from the above references that $\mathcal{T}$ is an algebra. More precisely, it is proved in [49, Thm. 0.3 Rem. 0.5] that there are bidifferential operators C_r such that for $f, g \in \mathcal{A}$,

$$T_{f,p} \circ T_{g,p} = \sum_{r=0}^{+\infty} p^{-r} T_{C_r(f,g),p} + O(p^{-\infty}). \tag{3.4.38}$$

We now give the precise formula for C_0 and C_1 computed in [49]. Let ∇^η be the Chern connection of (η, h^η). We denote again by ∇^η the induced connexion on $\mathrm{End}(\eta)$, and we decompose ∇^η according to the bigraduation:

$$\nabla^\eta = \nabla^{1,0} + \bar{\partial}^\eta. \tag{3.4.39}$$

Let

$$\langle \cdot , \cdot \rangle_{\Theta^Y} : \Omega^\bullet(Y, \mathrm{End}(\eta)) \times \Omega^\bullet(Y, \mathrm{End}(\eta)) \to \mathscr{C}^\infty(Y, \mathrm{End}(\eta)) \tag{3.4.40}$$

be the $\mathbb{C}$-bilinear pairing define by $\langle \alpha \otimes f, \beta \otimes g \rangle_{\Theta^Y} = \langle \alpha, \beta \rangle fg$ for $\alpha, \beta \in \Omega^\bullet(Y)$ and $f, g \in \mathcal{A}$.

Then we have:

$$\begin{aligned} C_0(f,g) &= fg, \\ C_1(f,g) &= -\frac{1}{2\pi} \langle \nabla^{1,0} f, \bar{\partial}^\eta g \rangle_{\Theta^Y}. \end{aligned} \tag{3.4.41}$$

For $f, g \in \mathcal{A}$, set

$$\{f, g\} = \frac{1}{2\pi\sqrt{-1}} \big(\langle \nabla^{1,0} g, \bar{\partial}^\eta f \rangle_{\Theta^Y} - \langle \nabla^{1,0} f, \bar{\partial}^\eta g \rangle_{\Theta^Y} \big). \tag{3.4.42}$$

Note that if $\eta = \mathbb{C}$, then $\{f, g\}$ is just the Poisson bracket of f and g associated with the symplectic form $2\pi\Theta^Y$.

By (3.4.41), we have in particular,

$$\begin{aligned}
&T_{f,p} \circ T_{g,p} = T_{fg,p} + O(p^{-1}) \\
&[T_{f,p}, T_{g,p}] = T_{[f,g],p} + \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + O(p^{-2}), \\
&[T_{f,p}, T_{g,p}]_+ = T_{[f,g]_+,p} - \frac{1}{p} T_{\langle \nabla^\eta f, \nabla^\eta g\rangle_{\Theta^Y},p} + O(p^{-2}),
\end{aligned} \tag{3.4.43}$$

where $[\,\cdot\,,\cdot\,]_+$ denotes the anti-commutator.

3.4.3 Infinite dimensional bundles.

From now on, we will consider $\mathcal{A}$ and $\mathcal{T}$ as infinite dimensional bundles of algebra on M: for $m \in M$,

$$\begin{aligned}
&\mathcal{A}_m = \mathscr{C}^\infty(Y_m, \mathrm{End}(\eta|_{Y_m})), \\
&\mathcal{T}_m = \{\text{Toeplitz operators on the fiber } Y_m\}.
\end{aligned} \tag{3.4.44}$$

In particular, an element of $\mathcal{T}$ define a family of elements of $\mathrm{End}(F_p)$, $p \in \mathbb{N}$. Moreover, $\|\cdot\|_{\mathscr{C}^0}$ defines a metric on the bundle $\mathcal{A}$, and $||\cdot||_\infty$ and $||\cdot||_1$ define two metrics on the bundle $\mathcal{T}$.

In the sequel, for any hermitian bundle $(\mathcal{V}, h^{\mathcal{V}})$ on M, we will still denote by $||\cdot||_\infty$ and $||\cdot||_1$ the induced metrics on $\mathcal{V} \otimes \mathcal{T}$.

We define a connection on $\mathcal{A}$ as follows: if $f \in \mathscr{C}^\infty(M, \mathcal{A}) = \mathscr{C}^\infty(N, \mathrm{End}(\eta))$ and $U \in T_\mathbb{R}M$, then

$$\nabla^{\mathcal{A}}_U f = \nabla^\eta_{U^H} f, \tag{3.4.45}$$

where U^H is the horizontal lift of U in $T^H_{M,\mathbb{R}}N$ (see (3.4.2)).

Define also $\mathcal{F}_p$ as the infinite dimensional bundle:

$$\mathcal{F}_{p,m} = \mathscr{C}^\infty(Y_m, (\eta \otimes L^p)|_{Y_m}). \tag{3.4.46}$$

Then F_p is a sub-bundle of $\mathcal{F}_p$ and $\mathcal{F}_p$ is endowed with the connection $\nabla^{\mathcal{F}_p}$ defined by

$$\nabla^{\mathcal{F}_p}_U s = \nabla^{\eta \otimes L^p}_{U^H} s, \tag{3.4.47}$$

where U^H is the horizontal lift of $U \in T_\mathbb{R}M$ in $T^H_{M,\mathbb{R}}N$.

Finally, $\mathcal{A}$ and $\mathcal{F}_p$ are equipped with the L^2 metrics $h^{\mathcal{A}}$ and $h^{\mathcal{F}_p}$ associated to $g^{T_\mathbb{R}Y}$, h^η and h^L. By Remark 3.1.6 and [11, Thm. 1.5], we know that $\nabla^{\mathcal{A}}$ and $\nabla^{\mathcal{F}_p}$ preserve the metrics $h^{\mathcal{A}}$ and $h^{\mathcal{F}_p}$. Furthermore, if ∇^{F_p} is the Chern connection on (F_p, h^{F_p}), then by (3.2.10) and (3.2.55), we have

$$\nabla^{F_p} = P_p \nabla^{\mathcal{F}_p} P_p. \tag{3.4.48}$$

Let R^{F_p} be the curvature of ∇^{F_p}. We denote again by P_p the projection from $\Lambda^\bullet(T^*_\mathbb{R}M) \otimes \mathcal{F}_p$ onto $\Lambda^\bullet(T^*_\mathbb{R}M) \otimes F_p$. The following theorem of Ma-Zhang [50, Thm 2.1] is the cornerstone of our approach.

Theorem 3.4.1. *Let $f \in \mathscr{C}^\infty(M, \mathcal{A})$. The forms $\nabla^{F_p} T_{f,p}$ and $\frac{1}{p} R^{F_p}$ are Toeplitz operator valued form, which means that there are $\varphi_r(f) \in \mathscr{C}^\infty(M, T^*_\mathbb{R}M \otimes \mathcal{A})$ and $R_r \in$*

$\mathscr{C}^\infty(M, \Lambda^2(T^*_{\mathbb{R}}M) \otimes \mathcal{A})$ *such that*

$$\begin{aligned} \nabla^{F_p} T_{f,p} &= \sum_{r=0}^{+\infty} T_{\varphi_r(f),p} p^{-r} + O(p^{-\infty}), \\ \frac{1}{p} R^{F_p} &= \sum_{r=0}^{+\infty} T_{R_r,p} p^{-r} + O(p^{-\infty}). \end{aligned} \tag{3.4.49}$$

Moreover, For $U, V \in T_{\mathbb{R}}M$, we have

$$\begin{aligned} &\varphi_0(f)(U) = \nabla^\eta_{U^H} f, \\ &R_0(U,V) = R^L(U^H, V^H). \end{aligned} \tag{3.4.50}$$

Recall the Lichnerowicz formula (3.2.34):

$$\begin{aligned} B^2_{p,u} = &-\frac{u}{2}(\nabla^p_{u,e_i})^2 + \frac{uK^X}{8} + \frac{u}{4} c(e_i)c(e_j)(L'^{\xi}_{i,j} + R^{F_p}_{i,j}) + \sqrt{\frac{u}{2}} c(e_i) f^\alpha (L'^{\xi}_{i,\alpha} + R^{F_p}_{i,\alpha}) \\ &+ \frac{f^\alpha f^\beta}{2}(L'^{\xi}_{\alpha,\beta} + R^{F_p}_{\alpha,\beta}) - u\psi_{1/\sqrt{u}} \Big(\bar{\partial}^M \partial^M i\omega\Big)^c \psi_{\sqrt{u}} - \frac{u}{16} \Big\| \Big(\bar{\partial}^X - \partial^X\Big) i\omega^X \Big\|^2_{\Lambda^\bullet(T^*_{\mathbb{R}}X)}. \end{aligned} \tag{3.4.51}$$

Using this formula and Theorem 3.4.1, we deduce that for $b \in B$,

$$B^2_{p,u}|_{X_b} \in \mathrm{Op}(X_b) \otimes \Lambda^\bullet(T^*_b B) \otimes \mathrm{End}\left(\Lambda^{0,\bullet}(T^*X_b) \otimes \xi|_{X_b}\right) \otimes \mathbb{C}[p] \otimes \mathcal{T}|_{X_b}, \tag{3.4.52}$$

where $\mathrm{Op}(X_b)$ is the algebra of scalar differential operators on X_b.

3.4.4 Operators with Toeplitz coefficients.

In this section, we extend the results of [46, Sects. 7.2-7.4] to the case of Toeplitz operators with value in the algebra of bounded operator on a fixed Hilbert space. We use the notations of Sections 3.4.2 and 3.4.3, and we work on a single fiber Y_m, which will be simply denoted by Y.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. We denote again by P_p the orthogonal projection

$$P_p \otimes \mathrm{Id}_{\mathcal{H}} \colon L^2(Y, L^p \otimes \eta) \otimes \mathcal{H} = L^2(Y, L^p \otimes \eta \otimes \mathcal{H}) \to H^0(Y, L^p \otimes \eta) \otimes \mathcal{H}, \tag{3.4.53}$$

and for every smooth family $A(y) \in \mathrm{End}(\eta_y) \otimes \mathcal{B}(\mathcal{H})$, $y \in Y$, we can define the operator

$$T_{A,p} = P_p A(\cdot) P_p \colon L^2(Y, L^p \otimes \eta \otimes \mathcal{H}) \to L^2(Y, L^p \otimes \eta \otimes \mathcal{H}). \tag{3.4.54}$$

Here again, we denote by $\|\cdot\|_\infty$ the operator norm for bounded operators acting on the Hilbert space $L^2(Y, L^p \otimes \eta \otimes \mathcal{H})$.

We extend the definition of Toeplitz operators to this situation: here again we call Toeplitz operator a family of operators $T_p \in \mathrm{End}(L^2(Y, L^p \otimes \eta \otimes \mathcal{H}))$ satisfying the two properties of Definition 3.1.8, with $f_r \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta) \otimes \mathcal{B}(\mathcal{H}))$.

Lemma 3.4.2. *The operator $T_{A,p}$ has a smooth Schwartz kernel*

$$T_{A,p}(y,y') \in (L^p \otimes \eta)_y \otimes (L^p \otimes \eta)^*_{y'} \otimes \mathcal{B}(\mathcal{H}) \tag{3.4.55}$$

with respect to $dv_Y(y')$.

For $\varepsilon > 0$, $\ell, m \in \mathbb{N}$, there is $C_{\ell,m,\varepsilon} > 0$ such that for all $p \geq 1$ and $y, y' \in Y$ with $d(y,y') > \varepsilon$,

$$\|T_{A,p}(y,y')\|_{\mathscr{C}^m(Y\times Y)} \leq C'_{\ell,m,\varepsilon} p^{-\ell}, \tag{3.4.56}$$

where the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^η, the usual derivation on $\mathcal{H}$ and h^L, h^η, $\|\cdot\|_{\mathcal{H}}$.

Proof. We denote by $P_p(y,y')$ the smooth Schwartz kernel of P_p. Then

$$T_{A,p}(y,y') = \int_Y P_p(y,y'')A(y'')P_p(y'',y)dv_Y(y''). \tag{3.4.57}$$

Let $\varepsilon > 0$ and $\ell, m \in \mathbb{N}$. Let $y, y' \in Y$ such that $d(y,y') > \varepsilon$. By [46, Prop. 4.1.5] we know that there is $C'_{\ell,m,\varepsilon} > 0$ such that

$$|P_p(y,y')|_{\mathscr{C}^m} \leq C'_{\ell,m,\varepsilon} p^{-\ell}, \tag{3.4.58}$$

and by [46, Thm. 4.2.1] we know that there are a constants $C_m > 0$ and $M_m > 0$ such that

$$|P_p(\cdot,\cdot)|_{\mathscr{C}^m(Y\times Y)} \leq C_m p^{M_m}. \tag{3.4.59}$$

Using this two facts, the uniform boundedness of $A(y'')$, $y'' \in Y$, and (3.4.57), we complete the proof of Lemma 3.4.2. $\square$

Recall that TY is endowed with the Hermitian structure induced by $R^L|_{TY\times TY}$. For $y_0 \in Y$, we choose $\{v_i\}_{i=1}^{n_Y}$ an orthonormal basis of $T_{y_0}Y$. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(v_j + \overline{v}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(v_j - \overline{v}_j)$, $j = 1, \dots, n_Y$, forms an orthonormal basis of $T_{\mathbb{R},y_0}Y$, which gives use an isomorphism $T_{\mathbb{R},y_0}Y \simeq \mathbb{R}^{2n_Y}$. We denote the dependence on the base point y_0 by adding a superscript y_0.

On $\mathbb{R}^{2n_Y} \simeq \mathbb{C}^{n_Y}$, we denote the coordinates by $(W_1, \dots, W_{2n_Y})$ or $(w_1, \dots, w_{n_Y})$, with $w_j = W_{2j-1} + \sqrt{-1}W_{2j}$. Let $\mathscr{P}$ be the operator on $L^2(\mathbb{R}^{2n_Y})$ defined by its kernel with respect to dW:

$$\mathscr{P}(W,W') = \frac{1}{(2\pi)^m} \exp\left(-\frac{1}{4}(|w|^2 + |w'|^2 - 2w\cdot w')\right). \tag{3.4.60}$$

Then $\mathscr{P}$ is the usual Bergman kernel on $\mathbb{C}^{n_Y}$.

We fix $y_0 \in Y$. As usually, for $\varepsilon > 0$ small enough, we identify the geodesic ball $B^Y(y_0, 4\varepsilon)$ with the ball $B^{T_{\mathbb{R},y_0}Y}(0,4\varepsilon)$ in $T_{\mathbb{R},y_0}Y$ via the exponential map. The various bundles appearing here on $B^{T_{\mathbb{R},y_0}Y}(0,4\varepsilon)$ are trivialized by mean of orthonormal frames at y_0 and of parallel transport for the corresponding connections along the rays $u \in [0,1] \mapsto uW$. Let $dv_{T_\mathbb{R}Y}$ be the volume form on $(T_{\mathbb{R},y_0}Y, g^{T_{\mathbb{R},y_0}Y})$, we denote by τ_{y_0} the function satisfying

$$dv_Y(W) = \tau_{y_0}(W)dv_{T_\mathbb{R}Y}(W), \quad \tau_{y_0}(0) = 1. \tag{3.4.61}$$

Let pr_Y be the natural projection from the fiberwise product $T_\mathbb{R}Y \times_Y T_\mathbb{R}Y$ to Y. Consider an operator $\Xi_p\colon L^2(Y, L^p\otimes\eta)\otimes\mathcal{H} \to L^2(Y, L^p\otimes\eta)\otimes\mathcal{H}$ which as a smooth kernel $\Xi_p(y,y')$ with respect to $dv_Y(y')$. Under our trivialization, this kernel induces a smooth section $\Xi_p^{y_0}(Z,Z')$ of $\mathrm{pr}_Y^*(\mathrm{End}\eta)\otimes\mathcal{B}(\mathcal{H})$ over $\{(y,W,W') : |W|,|W'| \leq 4\varepsilon\} \subset T_\mathbb{R}Y\times_Y T_\mathbb{R}Y$.

Let $Q_{r,y_0} \in \mathrm{End}(\eta_{y_0}) \otimes \mathcal{B}(\mathcal{H})[W,W']$, $r \in \mathbb{N}$, be polynomials in W, W' with values in $\mathrm{End}(\eta_{y_0})$ which depends smoothly on $y_0 \in Y$. In the sequel, we denote

$$p^{-n_Y} \Xi_p^{y_0}(W,W') \cong \sum_{r=0}^{k} (Q_{r,y_0}\mathscr{P})(\sqrt{p}W, \sqrt{p}W')p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}) \tag{3.4.62}$$

if there exist $0 < \varepsilon' < 4\varepsilon$ and $C_0 > 0$ such that for any $\ell \in \mathbb{N}$, there exist $C_{k,\ell}, M > 0$ such that for any $W, W' \in T_{\mathbb{R},y_0}Y$, $|W|, |W'| < \varepsilon'$ and any p, we have

$$\begin{aligned}\Bigg\| p^{-n_Y} \Xi_p^{y_0}(W,W')\tau_{y_0}^{1/2}(W)\tau_{y_0}^{1/2}(W') - \sum_{r=0}^{k} (Q_{r,y_0}(A)\mathscr{P})(\sqrt{p}W, \sqrt{p}W')p^{-\frac{r}{2}} \Bigg\|_{\mathscr{C}^\ell(Y)} \\ \leq C_{k,\ell}p^{-\frac{k+1}{2}}(1+\sqrt{p}|W|+\sqrt{p}|W'|)^M e^{-\sqrt{C_0 p}|W-W'|} + O(p^{-\infty}).\end{aligned} \tag{3.4.63}$$

Here, $\mathscr{C}^\ell(Y)$ denotes the $\mathscr{C}^\ell$-norm for the parameter $y_0 \in Y$ induced by the operator norms on $\mathrm{End}(\eta_{y_0})$ and $\mathcal{B}(\mathcal{H})$, and by $O(p^{-\infty})$ we mean a term such that for any $\ell, \ell_1 \in \mathbb{N}$, there exists $C_{\ell,\ell_1} > 0$ such that its $\mathscr{C}^{\ell_1}$-norm is dominated by $C_{\ell,\ell_1}p^{-\ell}$.

Recall that by [46, Lem. 7.2.3], there exist $J_{r,y_0} \in \mathrm{End}(\eta_{y_0})[W,W']$ polynomials in W, W' with values in $\mathrm{End}(\eta_{y_0})$ with the same parity as r and with

$$J_{0,y_0} = \mathrm{Id}_{\eta_{y_0}}, \tag{3.4.64}$$

such that

$$p^{-n_Y} P_p^{y_0}(W,W') \cong \sum_{r=0}^{k} (J_{r,y_0}\mathscr{P})(\sqrt{p}W, \sqrt{p}W')p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}) \tag{3.4.65}$$

Lemma 3.4.3. *Let $A \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta) \otimes \mathcal{B}(\mathcal{H}))$. Then there exist a family of $\mathrm{End}(\eta_{y_0}) \otimes \mathcal{B}(\mathcal{H})$-valued polynomials $\{Q_{r,y_0}(A)\}_{r\in\mathbb{N}, y_0\in Y}$ with the same parity as r and smooth in $y_0 \in Y$ such that for any $k \in \mathbb{N}$, $|Z|, |Z'| < \varepsilon/2$,*

$$p^{-n_Y} T_{A,p}^{y_0}(W,W') \cong \sum_{r=0}^{k} (Q_{r,y_0}(A)\mathscr{P})(\sqrt{p}W, \sqrt{p}W')p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}), \tag{3.4.66}$$

and moreover,

$$Q_{0,y_0}(A) = A(y_0). \tag{3.4.67}$$

Proof. In the above trivialization, any $B \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta) \otimes B(\mathcal{H}))$ induced a family with parameter $y_0 \in Y$ of functions $B^{y_0}(W) \in \mathscr{C}^\infty\left(B^{T_{y_0}Y}(0,4\varepsilon), \mathcal{H}\right)$. In the same way, the kernel $P_p(y,y')$ (resp. $T_{A,p}(y,y')$) induces a family of $\mathrm{End}(\eta_{y_0})$-valued (resp. $\mathrm{End}(\eta_{y_0}) \otimes \mathcal{B}(\mathcal{H})$-valued) kernels $P_p^{y_0}(W,W')$ (resp. $T_{A,p}^{y_0}(W,W')$) on $B^{T_{\mathbb{R},y_0}Y}(0,4\varepsilon) \times B^{T_{\mathbb{R},y_0}Y}(0,4\varepsilon)$.

Let $\rho\colon \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = \begin{cases} 1 \text{ for } |v| < 2, \\ 0 \text{ for } |v| > 4. \end{cases} \tag{3.4.68}$$

Thanks to (3.4.57) and (3.4.56), we know that $T_{A,p}^{y_0}(W,W')$, for $|W|, |W'| < \varepsilon/2$ in $T_{\mathbb{R},y_0}Y$, is determined up to term of order $\mathcal{O}(p^{-\infty})$ by the restriction of A to the ball $B^Y(y_0,\varepsilon)$. Thus,

$$\begin{aligned}T_{A,p}^{y_0}(W,W') = \int_{T_{\mathbb{R},y_0}Y} P_p^{y_0}(W,W'')\rho(2|W''|/\varepsilon)A^{y_0}(W'')P_p^{y_0}(W'',W') \\ \times \tau_{y_0}(W'')dv_{T_\mathbb{R}Y}(W'') + O(p^{-\infty}).\end{aligned} \tag{3.4.69}$$

We write the Taylor expansion of A^{y_0} as:

$$A^{y_0}(W) = \sum_{|\alpha|\le k} p^{-\frac{\alpha}{2}} \frac{\partial^\alpha A^{y_0}}{\partial W^\alpha}(0) \frac{(\sqrt{p}W)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}} O(|\sqrt{p}W|^{k+1}). \tag{3.4.70}$$

Multiplying this expansion with the one of $P_p^{y_0}(W, W')$ given in (3.4.65), we get the asymptotic expansion of

$$\tau_{y_0}^{1/2}(W) P_p^{y_0}(W, W'')(\tau_{y_0} A^{y_0})(W'') P_p^{y_0}(W'', W') \tau_{y_0}^{1/2}(W'). \tag{3.4.71}$$

We substitute this expansion in (3.4.69) and we integrate on $T_{\mathbb{R},y_0}Y$ after having done the change of variable $\sqrt{p}W'' \leftrightarrow W''$. Using moreover (3.4.64), this gives (3.4.66) and (3.4.67). $\square$

We now state the analogue of [46, Thm 7.3.1], which gives a criterion for being a Toeplitz operator.

Theorem 3.4.4. *Let $T_p \colon L^2(Y, L^p \otimes \eta \otimes \mathcal{H}) \to L^2(Y, L^p \otimes \eta \otimes \mathcal{H})$ be a family of bounded linear operators which satisfies the following three conditions:*

(i) *for any $p \in \mathbb{N}$, $P_p T_p P_p = T_p$;*

(ii) *for any $\varepsilon_0 > 0$ and $\ell, m \in \mathbb{N}$, there exists $C_{\ell,m} > 0$ such tat for all $p \ge 1$ and all $y, y' \in Y$ with $d(y, y') > \varepsilon_0$,*

$$\|T_p(y, y')\|_{\mathscr{C}^m(Y\times Y)} \le C_{\ell,m} p^{-\ell}; \tag{3.4.72}$$

(iii) *there exists a family of polynomial $\mathcal{Q}_{r,y_0} \in \mathrm{End}(\eta_{y_0}) \otimes \mathcal{B}(\mathcal{H})[W, W']$ with the same parity as r and depending smoothly in y_0 such that in the sense of* (3.4.62) *and* (3.4.63),

$$p^{-n_Y} T_p^{y_0}(W, W') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,y_0} \mathscr{P})(\sqrt{p}W, \sqrt{p}W') p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}). \tag{3.4.73}$$

Then $\{T_p\}_{p\ge 1}$ is a Toeplitz operator.

Proof. The proof is word for word the proof of [46, Thm 7.3.1], replacing therein $\mathrm{End}(E_{x_0})$ by $\mathrm{End}(\eta_{y_0}) \otimes \mathcal{B}(\mathcal{H})$ endowed with the operator norm. $\square$

Theorem 3.4.5. *For any $A, B \in \mathscr{C}^\infty(Y, \mathrm{End}(\eta) \otimes \mathcal{B}(\mathcal{H}))$, the product of $T_{A,p}$ and $T_{B,p}$ is a Toeplitz operator. More precisely, there are bidifferential operators C_r such that in the sense of* (3.4.37),

$$T_{A,p} T_{B,p} = \sum_{r=0}^{+\infty} p^{-r} T_{C_r(A,B),p} + O(p^{-\infty}), \tag{3.4.74}$$

and we have

$$C_0(A, B) = AB. \tag{3.4.75}$$

Proof. First, by [46, Lem. 7.1.1], if $F, G \in \mathbb{C}[W, W']$, there exist polynomials $\mathscr{K}[F, G] \in \mathbb{C}[W, W']$ such that

$$((F\mathscr{P}) \circ (G\mathscr{P}))(W, W') = \mathscr{K}[F, G](W, W') \mathscr{P}(W, W'). \tag{3.4.76}$$

As in (3.4.69), for $|W|, |W'| < \varepsilon/4$ in $T_{\mathbb{R},y_0}Y$, we have

$$(T_{A,p}T_{B,p})_{y_0}(W,W') = \int_{T_{\mathbb{R},y_0}Y} T^{y_0}_{A,p}(W,W'')\rho(4|W''|/\varepsilon)T^{y_0}_{B,p}(W'',W') \\ \times \tau_{y_0}(W'')dv_{T_{\mathbb{R}}Y}(W'') + O(p^{-\infty}). \quad (3.4.77)$$

As in the proof of (3.4.66), we find by multiplying the expansions of $T^{y_0}_{A,p}(W,W')$ and $T^{y_0}_{b,p}(W,W')$ given by (3.4.66) that

$$p^{-n_Y}(T_{A,p}T_{B,p})^{y_0}(W,W') \cong \sum_{r=0}^{k} (Q_{r,y_0}(A,B)\mathscr{P}_{y_0})(\sqrt{p}W,\sqrt{p}W')p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}), \quad (3.4.78)$$

and moreover,

$$Q_{r,y_0}(A,B) = \sum_{s+t=r} \mathscr{K}[Q_{s,y_0}(A), Q_{t,y_0}(B)]. \quad (3.4.79)$$

By Theorem 3.4.4,we know that $T_{A,p}T_{B,p}$ is a Toeplitz operator. Moreover, it follows from the proofs of Lemma 3.4.3 and Theorem 3.4.4 that C_r are bidifferential operators.

Finally, by (3.4.67), (3.4.76) and (3.4.79) we find

$$Q_{0,y_0}(A,B) = \mathscr{K}[Q_{0,y_0}(A), Q_{0,y_0}(B)] = A(y_0)B(y_0). \quad (3.4.80)$$

The proof of Theorem 3.4.5 is complete. □

3.4.5 Localization.

Fix $b_0 \in B$. We use the same notations and trivializations that in Section 3.3.1, except that we change therein L^p by F_p, so that now

$$\begin{aligned} \mathbb{E}_p &= \Lambda^{\bullet}_{b_0}(T^*_{\mathbb{R}}B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi \otimes F_p\right), \\ \mathbb{E} &= \Lambda^{\bullet}_{b_0}(T^*_{\mathbb{R}}B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi\right). \end{aligned} \quad (3.4.81)$$

Once again, we want to emphasize that the curtail difference with Section 3.3 is that the dimension of $\mathbb{E}_p$ is not constant but grows to infinity. This is why we have to use the operator norm on $\mathrm{End}(F_p)$ and Toeplitz operators (notably their boundedness and the properties of their derivatives).

We first prove that Lemma 3.3.1 still holds in the present situation.

Lemma 3.4.6. *For any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $p \in \mathbb{N}$, $u > 0$ and $s \in \boldsymbol{H}^{2k+2}(X, \mathbb{E}_p)$,*

$$||s||^2_{\boldsymbol{H}^{2k+2}(p)} \leq C_k p^{4k+4} \sum_{j=0}^{k+1} p^{-4j}||B_p^{2j}s||_{L^2}. \quad (3.4.82)$$

Proof. As in the proof of Lemma 3.3.1, we work locally on one of the U_{x_j}'s and trivialize $\mathbb{E}_p$ in the way indicated at the beginning of Section 3.3.1.

Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to $\nabla^{T_{\mathbb{R}}X}$ along the curve $t \in [0,1] \mapsto tZ$. Let Γ^{ξ}, Γ^{F_p} and $\Gamma^{\Lambda^{0,\bullet},LC}$ be the connection form of ∇^{ξ}, ∇^{F_p} and $\nabla^{\Lambda^{0,\bullet},LC}$ with respect to any fixed frame for ξ, F_p and $\Lambda^{0,\bullet}(T^*X)$ which is parallel along the curve $t \in [0,1] \mapsto tZ$ under the trivialization on U_{x_k}.

Then

$$\nabla^p_{1,\widetilde{e_i}} = \nabla_{\widetilde{e_i}} + (\Gamma^{\Lambda^{0,\bullet},LC} + \Gamma^{\xi} + \Gamma^{F_p})(\widetilde{e_i}) + \frac{1}{\sqrt{2}} S(\widetilde{e_i}, \widetilde{e_j}, f_\alpha) c(\widetilde{e_j}) f^\alpha \\ + \frac{1}{2} S(\widetilde{e_i}, f_\alpha, f_\beta) f^\alpha f^\beta + \frac{1}{2}\Big(i_{\widetilde{e_i}}\left(\bar{\partial}^M - \partial^M\right) i\omega\Big)^c. \quad (3.4.83)$$

Moreover, we know that the Lie derivative $\mathcal{L}_Z \Gamma^{F_p}$ of Γ^{F_p} is given by $\mathcal{L}_Z \Gamma^{F_p} = i_Z R^{F_p}$ (see [46, (1.2.32)] for instance). Similarly, $\mathcal{L}_Z \Gamma^L = i_Z R^L$. This, together with Theorem 3.4.1, implies that Γ^{F_p} is a Toeplitz operator and that there is a $\Gamma \in \mathscr{C}^\infty(Z_{b_0}, T^*_{\mathbb{R}} N \otimes \mathbb{C})$ such that

$$\Gamma^{F_p}(U) = p T_{\Gamma(U^H),p} + O(1). \quad (3.4.84)$$

Hence, (3.4.83) become

$$\nabla^p_{u,e_i} = \nabla_{e_i} + \Gamma^{\Lambda^{0,\bullet},LC} + \Gamma^{\xi} + p T_{\Gamma(e_i^H),p} + \frac{1}{\sqrt{2u}} S_{i,j,\alpha} c(e_j) f^\alpha + \frac{1}{2u} S_{i,\alpha,\beta} f^\alpha f^\beta \\ + \frac{1}{2}\psi_{1/\sqrt{u}}\Big(i_{e_i}\left(\bar{\partial}^M - \partial^M\right) i\omega\Big)^c \psi_{\sqrt{u}} + O(1). \quad (3.4.85)$$

We now prove that B_p^2 has a similar structure as in (3.3.8). By (3.4.35), we know that for $s \in \boldsymbol{H}^1(U_{x_j}, \mathbb{E}_{p,x_j})$,

$$||T_{\Gamma(e_i^H),p} s||_{L^2} \leq C||s||_{L^2} \quad \text{and} \quad ||T_{\Gamma^H,p} \nabla_U s||_{L^2} \leq C||s||_{\boldsymbol{H}^1(p)}. \quad (3.4.86)$$

Moreover, using (3.4.43), Theorem 3.4.1 and (3.4.84), we find

$$\begin{aligned} \nabla_U T_{\Gamma(e_i^H),p} &= \nabla_U^{F_p} T_{\Gamma(e_i^H),p} - \Gamma^{F_p}(U) T_{\Gamma(e_i^H),p} \\ &= T_{\Gamma(e_i^H),p} \nabla_U^{F_p} + T_{U^H(\Gamma(e_i^H)),p} - \Gamma^{F_p}(U) T_{\Gamma(e_i^H),p} + O(p^{-1}) \\ &= T_{\Gamma(e_i^H),p} \nabla_U + T_{U^H(\Gamma(e_i^H)),p} - [\Gamma^{F_p}(U), T_{\Gamma(e_i^H),p}] + O(p^{-1}) \\ &= T_{\Gamma(e_i^H),p} \nabla_U + T_{U^H(\Gamma(e_i^H)),p} - p[T_{\Gamma(U^H),p}, T_{\Gamma(e_i^H),p}] + O(1) \\ &= T_{\Gamma(e_i^H),p} \nabla_U + T_{U^H(\Gamma(e_i^H)),p} + O(1), \end{aligned} \quad (3.4.87)$$

where $O(p^{-1})$ and $O(1)$ denote operator of degree 0 acting on F_p. As a consequence, we have for $s \in \boldsymbol{H}^1(U_{x_j}, \mathbb{E}_{p,x_0})$,

$$||\nabla_U T_{\Gamma^H,p} s||_{L^2} \leq C||s||_{\boldsymbol{H}^1(p)}. \quad (3.4.88)$$

Let $D^X = \bar{\partial}^X + \bar{\partial}^{X,*}$ be the Dirac operator on $\Lambda^{0,\bullet}(T^*X) \otimes \xi$. Using (3.2.34), [46, Thm. 1.4.7], (3.4.85), (3.4.86) and (3.4.88), we find as in (3.3.8):

$$B_p^2 = D^{X,2} + R + p\mathcal{O}_{p,1} + p\mathcal{O}^1_{p,0} + p^2\mathcal{O}^2_{p,0} \quad (3.4.89)$$

where R is a differential operators acting on $\Lambda^\bullet_{b_0}(T^*_{\mathbb{R}}B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi\right)_{x_j}$, and $\mathcal{O}_{p,1}$, $\mathcal{O}^1_{p,0}$ and $\mathcal{O}^2_{p,0}$ are differential operators acting on $\mathbb{E}_{p,x_j}$ such that there is $C > 0$ satisfying for $s \in \boldsymbol{H}^{k+1}(U_{x_j}, \mathbb{E}_{p,x_0})$:

$$\begin{aligned} &||\mathcal{O}_{p,1} s||_{\boldsymbol{H}^k(p)} \leq C||s||_{\boldsymbol{H}^{k+1}(p)}, \\ &||\mathcal{O}^i_{p,0} s||_{\boldsymbol{H}^k(p)} \leq C||s||_{\boldsymbol{H}^k(p)}, \quad i = 1, 2. \end{aligned} \quad (3.4.90)$$

The proof of Lemma 3.4.83 follows from (3.4.89) and (3.4.90) exactly in the same way as Lemma 3.3.1 follows from (3.3.8). □

Now, we want to prove an analogue of Proposition 3.3.2. The main ingredient in the proof of this proposition is the spectral gap of the Dirac operator. Thus, we begin with the following lemma. Recall that D_p is the Dirac operator on $\Lambda^{0,\bullet}(T^*X)\otimes\xi\otimes F_p$.

Lemma 3.4.7. *There exist $C_0, C_L>0$ and $\mu_0>0$ such that*

$$\mathrm{Sp}(D_p^2)\subset\{0\}\cup]C_0p-C_L,+\infty[. \tag{3.4.91}$$

Proof. We recall first some classic estimates. The Lefschetz operator L on on $\Lambda^\bullet(T^*_{\mathbb{R}}X)$ is given by $L=(\omega^X\wedge)$, and its adjoint with respect to the metric on $\Lambda^\bullet(T^*_{\mathbb{R}}X)$ is denoted by Λ. Let

$$T=[\Lambda,\partial\omega] \tag{3.4.92}$$

be the *Hermitian torsion operator*.

Then Nakano's inequality (see [46, Thm. 1.4.14]) states that for $s\in\Omega^\bullet(X,F_p)$,

$$\frac{3}{2}\langle D_p^2s,s\rangle\geq\langle[\sqrt{-1}R^{F_p},\Lambda]s,s\rangle-\frac{1}{2}\left(||Ts||^2_{L^2}+||T^*s||^2_{L^2}+||\overline{T}s||^2_{L^2}+||\overline{T}^*s||^2_{L^2}\right). \tag{3.4.93}$$

Set

$$K_X^*=\Lambda^{(n,0)}(T^*X)=\det(TX)\quad\text{and}\quad\widetilde{F_p}=F_p\otimes K_X^*. \tag{3.4.94}$$

We can extend the natural isomorphism $K_X\otimes K_X^*\simeq\mathbb{C}$ to get an isometry

$$\Lambda^{(n,q)}(T^*X)\otimes\widetilde{F_p}\xrightarrow{\sim}\Lambda^{(0,q)}(T^*X)\otimes F_p. \tag{3.4.95}$$

Its inverse is denoted by $s\mapsto\tilde{s}$.

As done in [46, Lem. 1.4.17], we can apply Nakano's inequality to the bundle $\widetilde{F_p}$ and obtain that for $s\in\Omega^{(0,\bullet)}(X,F_p)$,

$$\frac{3}{2}\langle D_p^2s,s\rangle\geq\langle R^{F_p\otimes K_X^*}(w_j,\overline{w}_k)\overline{w}^k\wedge i_{\overline{w}_j}s,s\rangle-\frac{1}{2}\left(||T^*\tilde{s}||^2_{L^2}+||\overline{T}\tilde{s}||^2_{L^2}+||\overline{T}^*\tilde{s}||^2_{L^2}\right). \tag{3.4.96}$$

From Theorem 3.4.1 and (3.4.96) we get for $s\in\Omega^{(0,\bullet)}(X,F_p)$:

$$||D_ps||^2_{L^2}\geq\frac{2}{3}p\langle T_{R^L(w_j^H,\overline{w}_k^H),p}\overline{w}^k\wedge i_{\overline{w}_j}s,s\rangle-C||s||^2_{L^2}. \tag{3.4.97}$$

Fix $x\in X$ and $y\in Y_x$, the (1,1)-form $(U,V)\mapsto R^L_{(x,y)}(U^H,V^H)$ is positive on T_xX in the sense of Assumption 3.1.4. In particular, there is a orthonormal basis $\{w'_j\}$ of T_xX such that

$$R^L_{(x,y)}(w_j^H,\overline{w}_k^H)\overline{w}^k\wedge i_{\overline{w}_j}=a_j(x,y)\overline{w'}^j\wedge i_{\overline{w'}_j}, \tag{3.4.98}$$

with $a_j(x,y)>0$. Thus, there is a constant $C>0$ such that for $\sigma\in\Lambda^{(0,>0)}(T_x^*X)\otimes\mathcal{F}_{p,x}$,

$$\begin{aligned}\langle P_pR^L(w_j^H,\overline{w}_k^H)P_p\overline{w}^k\wedge i_{\overline{w}_j}\sigma,\sigma\rangle_{\Lambda^{(0,\bullet)}(T_x^*X)\otimes\mathcal{F}_{p,x}}&=\langle P_pa_j(x,\cdot)\overline{w'}^j\wedge i_{\overline{w'}_j}P_p\sigma,\sigma\rangle_{\Lambda^{(0,\bullet)}(T_x^*X)\otimes\mathcal{F}_{p,x}}\\&=\langle a_j(x,\cdot)\overline{w'}^j\wedge i_{\overline{w'}_j}P_p\sigma,P_p\sigma\rangle_{\Lambda^{(0,\bullet)}(T_x^*X)\otimes\mathcal{F}_{p,x}}\\&\geq C||P_p\sigma||^2_{\Lambda^{(0,\bullet)}(T_x^*X)\otimes\mathcal{F}_{p,x}}.\end{aligned} \tag{3.4.99}$$

From (3.4.35), (3.4.96) and (3.4.99) we deduce that here are $C_0,C_L>0$ such that for $s\in\Omega^{(0,>0)}(X,F_p)$,

$$||D_ps||^2_{L^2}\geq(C_0p-C_L)||s||^2_{L^2}. \tag{3.4.100}$$

Finally, if $s\in\Omega^{(0,0)}(X,F_p)$ satisfies $D_p^2s=\lambda s$ for some $\lambda\neq0$, then $0\neq D_ps\in\Omega^{(0,1)}(X,F_p)$ is still an eigenvector of D_p^2 for the eigenvalue λ, hence $\lambda\geq C_0p-C_L$. The proof of Lemma 3.4.7 is completed. ☐

Recall that the functions F_u, G_u and H_u and their tilded versions have been defined in (3.3.18) and (3.3.19).

We still denote by π the projection $\pi\colon X\times_B X\to B$ be the projection from the fiberwise product $X\times_B X$ to B. Then $\tilde{\mathrm{G}}_u(vB_p^2)(\cdot,\cdot)$ is a section of $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$ over $X\times_B X$. Let $\nabla^{\mathbb{E}_p}$ be the connection on $\mathbb{E}_p$ induced by $\nabla^{\Lambda^\bullet(T^*_\mathbb{R}B)}$, $\nabla^{\Lambda^{0,\bullet},LC}$, ∇^{F_p} and ∇^ξ, and let $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced connection on $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$. In the same way, let $h^{\mathbb{E}_p}$ be the metric on $\mathbb{E}_p$ induced by $h^{\Lambda^\bullet(T^*_\mathbb{R}B)}$, $h^{\Lambda^{0,\bullet},LC}$, h^L and h^ξ, and let $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced metric on $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$. Note that this metric restricts to the operator norm on the bundle $\mathrm{End}(\mathbb{E}_p)$ over $M\simeq\{(b,x,x')\in X\times_B X\,:\,x=x'\}$. We can now prove the analogue of Proposition 3.3.2:

Proposition 3.4.8. *For any $m\in\mathbb{N}$, $\varepsilon>0$, there exist $C>0$ and $N\in\mathbb{N}$ such that for any $u>0$ and any $p\in\mathbb{N}^*$,*

$$\left\|\tilde{\mathrm{G}}_{\frac{u}{p}}\Big(\frac{u}{p}B_p^2\Big)(\cdot,\cdot)\right\|_{\mathscr{C}^m}\leq Cp^N\exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{3.4.101}$$

Where the $\mathscr{C}^m$-norm is induced by $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$.

Proof. This proposition follows from Lemmas 3.4.6 and 3.4.7 exactly as Proposition 3.3.2 follows from Lemma 3.3.1 and (3.3.31). The only difference is that here we decompose B_p^2 as

$$\begin{aligned}&B_p^2=D_p^2+R_p,\\&R_p\in\Lambda^{\geq1}(T^*_\mathbb{R}B)\otimes\mathrm{Op}_X^{\leq1}(\Lambda^{0,\bullet}(T^*X)\otimes\xi)\otimes\mathbb{C}[p]\otimes\mathcal{T},\end{aligned} \tag{3.4.102}$$

and thus to obtain the analogues of (3.3.34) and (3.3.47), we also have to use the fact that Toeplitz operators are uniformly bounded for the operator norm (see (3.4.35)). □

Corollary 3.4.9. *For any $m\in\mathbb{N}$, $\varepsilon>0$, there exist $C(u)>0$ a rational fraction in $\sqrt{u}$ and $N\in\mathbb{N}$ such that for any $u>0$ and any $p\in\mathbb{N}^*$,*

$$\left\|\psi_{1/\sqrt{p}}\tilde{\mathrm{G}}_{\frac{u}{p}}(B^2_{p,u/p})(\cdot,\cdot)\right\|_{\mathscr{C}^m}\leq C(u)p^N\exp\left(-\frac{\varepsilon^2 p}{16u}\right). \tag{3.4.103}$$

Proof. As $B_{p,u}=\sqrt{u}\psi_{1/\sqrt{u}}B_p\psi_{\sqrt{u}}$, we have

$$\psi_{1/\sqrt{p}}\tilde{\mathrm{G}}_{\frac{u}{p}}(B^2_{p,u/p})\psi_{\sqrt{p}}=\psi_{1/\sqrt{u}}\tilde{\mathrm{G}}_{\frac{u}{p}}\left(\frac{u}{p}B_p^2\right)\psi_{\sqrt{u}}. \tag{3.4.104}$$

Thus, Corollary 3.4.9 follows from Proposition 3.4.8. □

3.4.6 Convergence of the heat kernel.

Here, we get the analogue of the results of Sections 3.3.2 and 3.3.3, and we prove Theorem 3.1.9. By comparison to Section 3.3, the difficulty is twofold. Firstly, as above in Section 3.4.5, we have to take into account the fact that the dimension of F_p grows to infinity, which is done thanks to Toeplitz operators. Secondly, if we can prove the convergence of the heat kernel of the rescaled operator to the heat kernel of some asymptotic operators in the vein Section 3.3.3, we can no longer compute the “limiting" heat kernel explicitly. However, using the results of Section 3.4.4, we can give the asymptotic of this heat kernel, which will enable us to conclude.

Fix $u > 0$, $b_0 \in B$ and $x_0 \in X_{b_0}$. We use the same notations and trivializations that in Section 3.3.2, changing therein L^p by F_p, and thus $p\Gamma^L$ by Γ^{F_p}. We get a connexion

$$\nabla^{\mathbb{E}_{p,x_0}} = \nabla + \rho(|Z|/\varepsilon)\left(\Gamma^{F_p} + \Gamma_1\right), \tag{3.4.105}$$

on the trivial bundle

$$\mathbb{E}_{p,x_0} = \Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \left(\Lambda^{0,\bullet}(T^*X) \otimes \xi \otimes F_p\right)_{x_0} \tag{3.4.106}$$

over $T_{x_0}X$, as well as a Laplacian $\Delta^{\mathbb{E}_{p,x_0}}$.

Recall that $\{f_\alpha\}$ denotes a frame of $T_\mathbb{R}B$, with dual frame $\{f^\alpha\}$. Let $\widetilde{e_i}(Z)$ be the parallel transport of e_i with respect to $\nabla^{T_\mathbb{R}X_0,LC}$ along the curve $t \in [0,1] \mapsto tZ$. Then $\{\widetilde{e_i}\}_i$ is an orthonormal frame of $T_\mathbb{R}X_0$.

Set

$$\begin{aligned}\Phi = \frac{K^X}{8} + \frac{1}{4}c(\widetilde{e_i})c(\widetilde{e_j})L'^{\xi}(\widetilde{e_i},\widetilde{e_j}) + \frac{1}{\sqrt{2}}c(\widetilde{e_i})f^\alpha L'^{\xi}(\widetilde{e_i},f_\alpha) + \frac{f^\alpha f^\beta}{2}L'^{\xi}(f_\alpha,f_\beta)\\ - \left(\bar\partial^M\partial^M i\omega\right)^c - \frac{1}{16}\left\|\left(\bar\partial^X - \partial^X\right)i\omega^X\right\|^2_{\Lambda^\bullet(T^*_\mathbb{R}X)}\end{aligned} \tag{3.4.107}$$

and

$$\begin{aligned}M_{p,x_0} = \frac{1}{2}\Delta^{\mathbb{E}_{p,x_0}} + \rho(|Z|/\varepsilon)\Phi\\ + \rho(|Z|/\varepsilon)\left(\frac{1}{4}c(\widetilde{e_i})c(\widetilde{e_j})R^{F_p}(\widetilde{e_i},\widetilde{e_j}) + \frac{1}{\sqrt{2}}c(\widetilde{e_i})f^\alpha R^{F_p}(\widetilde{e_i},f_\alpha) + \frac{f^\alpha f^\beta}{2}R^{F_p}(f_\alpha,f_\beta)\right).\end{aligned} \tag{3.4.108}$$

Then M_{p,x_0} is a second order elliptic differential operator acting on $\mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{p,x_0})$. Moreover, if $\mathcal{B}_{x_0}$ is the algebra:

$$\mathcal{B}_{x_0} = \mathrm{Op}(T_{\mathbb{R},x_0}X) \otimes \Lambda^\bullet(T^*_{b_0}B) \otimes \mathrm{End}\left(\Lambda^{0,\bullet}(T^*_{x_0}X) \otimes \xi_{x_0}\right) \otimes \mathbb{C}(\sqrt{p}) \otimes \mathcal{T}_{x_0}, \tag{3.4.109}$$

then Theorem 3.4.1, $\{(M_p)_Z\}_{p\geq 1}$ is in $\mathcal{B}_{x_0}$. Finally, near 0, $\nabla^{\mathbb{E}_{p,x_0}}$ equals ∇^p and M_{p,x_0} equals B_p^2.

Remark 3.4.10. Working on $\mathbb{E}_{p,x_0}$ amount to replace the fibration $Z \xrightarrow{Y} X$ by the trivial fibration $T_{\mathbb{R},x_0}X \times Y \to T_{\mathbb{R},x_0}X$. However, as pointed out earlier, we cannot substitute $\mathbb{E}_{p,x_0}$ here by some fixed $\mathbb{E}_{x_0}$ as in Section 3.3.2.

Let $\exp(-B_p^2)(Z,Z')$ and $\exp(-M_{p,x_0})(Z,Z')$ be the smooth heat kernel of B_p^2 and M_{p,x_0} with respect to $dv_{X_0}(Z')$.

Lemma 3.4.11. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $p \in \mathbb{N}^*$,*

$$\left\|\exp\left(-\frac{u}{p}B_p^2\right)(x_0,x_0) - \exp\left(-\frac{u}{p}M_{p,x_0}\right)(0,0)\right\|_{\mathscr{C}^m(M)} \leq Cp^N\exp(-\frac{\varepsilon^2 p}{16u}), \tag{3.4.110}$$

where $\|\cdot\|_{\mathscr{C}^m(M)}$ denotes the $\mathscr{C}^m$-norm in the parameters $b_0 \in B$ and $x_0 \in X$ induced by $\nabla^{\mathrm{End}(\mathbb{E}_p)}$ and the operator norm $h^{\mathrm{End}(\mathbb{E}_p)}$.

Proof. As explain in the proof of Lemma 3.3.7, we can prove Lemma 3.4.11 by proving analogs of Lemma 3.4.6 and Porposition 3.4.8 for M_{p,x_0}, and using the finite propagation speed of the wave equation. □

In the sequel, if $U \in T_{\mathbb{R}}M$, we denote by U^H its lift to $T^H_{M,\mathbb{R}}N$. Moreover, the basis $\{f_\alpha\}$ of $T_{\mathbb{R}}B$ has already been identified with a basis of $T^H_{\mathbb{R}}M$, and **when we write f^H_α we mean the lift in $T^H_{M,\mathbb{R}}N$ of f_α wiewed as an element of $T^H_{\mathbb{R}}M$** (which is not necessarily the same as the lift of $f_\alpha \in T_{\mathbb{R}}B$ in $T^H_{B,\mathbb{R}}N$). If e_{a_1}, e_{a_2} are some vectors among the e_i's and the f_α's we set

$$R^L_{a_1,a_2} = R^L(e^H_{a_1}, e^H_{a_2}). \tag{3.4.111}$$

To simplify the notations, we also write c^i for $c(e^H_i)$.

Similarly to what is done in (3.3.65), we define for $t = \frac{1}{\sqrt{p}}$, $s \in \mathscr{C}^\infty(T_{x_0}X, \mathbb{E}_{p,x_0})$ and $Z \in T_{x_0}X$:

$$\begin{aligned} &(S_t s)(Z) = s(Z/t),\\ &\nabla_t = tS_t^{-1}\kappa^{1/2}\nabla^{\mathbb{E}_{p,x_0}}\kappa^{-1/2}S_t,\\ &\mathscr{L}_t = t^2 S_t^{-1}\kappa^{1/2}M_{p,x_0}\kappa^{-1/2}S_t, \end{aligned} \tag{3.4.112}$$

Let $\Delta = \sum_i (\nabla_{e_i})^2$ be the ordinary Laplacian on $T_{\mathbb{R},x_0}X$, and recall that $[\cdot\,,\cdot]_+$ is our notation for the anti-commutator. Define for $U \in T_{\mathbb{R},x_0}X$:

$$\begin{aligned} \underline{\nabla}_{t,U} &= \nabla_U + T_{\frac{1}{2}R^L(Z^H,U^H),p}(x_0),\\ \underline{\mathscr{L}}_t &= -\frac{1}{2}\sum_i \Big\{{\nabla_{e_i}}^2 + [\nabla_{e_i}, T_{\frac{1}{2}R^L(Z^H,e^H_i),p}(x_0)]_+ + T_{(\frac{1}{2}R^L(Z^H,e^H_i))^2,p}(x_0)\Big\}\\ &\quad + T_{\frac{1}{4}c^ic^jR^L_{i,j} + \frac{1}{\sqrt{2}}c^if^\alpha R^L_{i,\alpha} + \frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta},p}(x_0). \end{aligned} \tag{3.4.113}$$

Proposition 3.4.12. *When $t \to 0$, we have the following asymptotic in $\mathcal{B}_{x_0}$*

$$\nabla_{t,e_i} = \underline{\nabla}_{t,e_i} + O(t) \ \text{and} \ \ \mathscr{L}_t = \underline{\mathscr{L}}_t + O(t). \tag{3.4.114}$$

Proof. First, by Theorem 3.4.1, if f is a smooth section of $\mathcal{A}$ over a compact subset of M, there is a $C > 0$ such that

$$||\nabla^{F_p}T_{f,p}||_\infty \leq C. \tag{3.4.115}$$

Let f be a section of $\mathcal{A}$. For $Z \in T_{\mathbb{R},x_0}X$, $|Z| \leq \varepsilon$, we denote by $f_Z \in \mathcal{A}_Z$ the restriction of f to the fibre Y_Z. Then $T_{f_Z,p}$ acts on $F_{p,Z} \simeq F_{p,x_0}$ and by the preceding inequality, we have

$$||T_{f_Z,p} - T_{f_0,p}||_\infty \leq C|Z|. \tag{3.4.116}$$

By (3.4.105) and (3.4.112), we have

$$\nabla_{t,e_i}(Z) = \kappa^{1/2}(tZ)\Big\{\nabla_{e_i} + \rho(t|Z|/\varepsilon)\Big(t\Gamma^{F_p}_{tZ}(e_i) + t\Gamma_{1,tZ}(e_i)\Big)\Big\}\kappa^{-1/2}(tZ). \tag{3.4.117}$$

Moreover, recall that (3.3.69) gives

$$t\Gamma_{1,tZ}(e_i) = O\left(t^2\right). \tag{3.4.118}$$

Finally, by (3.3.69) and (3.4.84), we know that

$$\begin{aligned} &\Gamma^{F_p}_Z(U) = pT_{\Gamma^L_Z(U^H),p} + O(1),\\ &t^{-1}\Gamma^L_{tZ}(e_i) = \frac{1}{2}R^L_{x_0}(Z, e_i) + O(t). \end{aligned} \tag{3.4.119}$$

Thus, by (3.3.68), (3.4.116), (3.4.117) and (3.4.118), and the fact that $\rho(0) = \kappa(0) = 1$, we find the first asymptotic development of Proposition 3.4.12.

As in (3.3.70) and (3.3.72), we have

$$\begin{aligned}\mathscr{L}_t = -g^{ij}(tZ)&\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TX_0}_{e_i}e_j}\right)\\ &+ t^2\rho(t|Z|/\varepsilon))\Big\{\kappa^{1/2}\Big(\Phi + \frac{1}{4}c(\widetilde{e_i})c(\widetilde{e_j})R^{F_p}(\widetilde{e_i},\widetilde{e_j})\\ &+\frac{1}{\sqrt{2}}c(\widetilde{e_i})f^\alpha R^{F_p}(\widetilde{e_i},f_\alpha) + \frac{f^\alpha f^\beta}{2}R^{F_p}(f_\alpha,f_\beta)\Big)\kappa^{-1/2}\Big\}_{tZ}. \end{aligned} \quad (3.4.120)$$

From Theorem 3.4.1, the first development in (3.4.114), (3.4.116) and (3.4.120), we find

$$\mathscr{L}_t = -\frac{1}{2}\sum_i\left(\underline{\nabla}_{t,e_i}\right)^2 + T_{\frac{1}{4}c^ic^jR^L_{i,j}+\frac{1}{\sqrt{2}}c^if^\alpha R^L_{i,\alpha}+\frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta},p}(x_0) + O(t). \quad (3.4.121)$$

Using the first equation of (3.4.43) and (3.4.121), we get the second identity of (3.4.114). The proof of Proposition 3.4.12 is completed. □

The next step is to prove an analogue of Theorem 3.3.21.

Let $e^{-\mathscr{L}_t}(Z,Z')$, $e^{-\underline{\mathscr{L}}_t}(Z,Z')$ be the smooth kernels of the operators $e^{-\mathscr{L}_t}$, $e^{-\underline{\mathscr{L}}_t}$ with respect to $dv_{TX}(Z')$. Let pr_X be the projection from the fiberwise product $T_\mathbb{R}X \times_X T_\mathbb{R}X$ to X, then these kernels are sections of $\mathrm{pr}_X^*(\mathrm{End}(\mathbb{E}_p))$ over $T_\mathbb{R}X \times_X T_\mathbb{R}X$.

Theorem 3.4.13. *For $u > 0$ fixed, there exists $C > 0$ such that for $t > 0$ and $Z, Z' \in T_{\mathbb{R},x_0}X$ with $|Z|, |Z'| \leq 1$, we have the following estimates form the operator norm:*

$$\left\|\left(e^{-u\mathscr{L}_t} - e^{-u\underline{\mathscr{L}}_t}\right)(Z,Z')\right\| \leq Ct^{1/(2n_X+1)}. \quad (3.4.122)$$

The proof of Theorem 3.4.13 follows the same strategy as the proof of Theorem 3.3.21 in Section 3.3.3. Here again, the difficulties coming from the fact that the dimension on F_p tend to infinity are dealt with the properties of Toeplitz operators.

Recall that we add a superscript (0) to the objects introduced above to denote their part of degree 0 in $\Lambda^\bullet(T^*_{\mathbb{R},b_0}B)$.

Let $||\cdot||_{t,0}$ be the L^2 norm on $\mathscr{C}^\infty(T_{\mathbb{R},x_0}X, \mathbb{E}_{p,x_0})$ induced by $h^{\Lambda^\bullet(T^*_\mathbb{R}B)}_{x_0}$, $h^{\Lambda^{0,\bullet}}_{x_0}$, $h^\xi_{x_0}$, $h^{F_p}_{x_0}$ and the volume form $dv_{TX}(Z)$. For $s \in \mathscr{C}^\infty(X_0, \mathbb{E}_{p,x_0})$, $m \in \mathbb{N}^*$, and $p \in \mathbb{N}^*$, set

$$\begin{aligned}||s||^2_{t,m} &= \sum_{\ell\leq m}\sum_{i_1,\dots,i_\ell}||\nabla^{(0)}_{t,e_{i_1}}\cdots\nabla^{(0)}_{t,e_{i_\ell}}s||^2_{t,0},\\ ||s||^2_{\underline{t},m} &= \sum_{\ell\leq m}\sum_{i_1,\dots,i_\ell}||\underline{\nabla}_{t,e_{i_1}}\cdots\underline{\nabla}_{t,e_{i_\ell}}s||^2_{\underline{t},0}.\end{aligned} \quad (3.4.123)$$

We denote by $\boldsymbol{H}^m_t$ the Sobolov space $\boldsymbol{H}^m(X_0, \mathbb{E}_{p,x_0})$ endowed with the norm $||\cdot||_{t,m}$, and by $\boldsymbol{H}^{-1}_t$ the Sobolev space of order -1 endowed with the norm

$$||s||_{t,-1} = \sup_{s'\in\boldsymbol{H}^1_p\setminus\{0\}}\frac{\langle s,s'\rangle_{p,0}}{||s'||_{t,1}}. \quad (3.4.124)$$

Finally, if $A \in \mathscr{L}(\boldsymbol{H}^k_t, \boldsymbol{H}^m_t)$, we denote by $||A||^{k,m}_t$ the operator norm of A associated with $||\cdot||_{t,k}$ and $||\cdot||_{t,m}$.

Let

$$\mathscr{R}_t = \mathscr{L}_t - \mathscr{L}^{(0)}_t. \quad (3.4.125)$$

Proposition 3.4.14. *There exist constants $C_1, C_2, C_3 > 0$ such that for any $t > 0$ and any $s, s' \in \mathscr{C}^\infty(X_0, \mathbb{E}_{p,x_0})$,*

$$\begin{aligned}
&\langle \mathscr{L}_t^{(0)} s, s\ \rangle_{t,0} \geq C_1 ||s||_{t,1}^2 - C_2 ||s||_{t,0}^2, \\
&\left| \langle \mathscr{L}_t^{(0)} s, s'\ \rangle_{t,0} \right| \leq C_3 ||s||_{t,1} ||s'||_{t,1}, \\
&\|\mathscr{R}_t s\|_{t,0} \leq C_4 ||s||_{t,1}.
\end{aligned} \tag{3.4.126}$$

Proof. By (3.4.120), we have a similar identity as in (3.3.80):

$$\langle \mathscr{L}_t^{(0)} s, s\ \rangle_{t,0} = \frac{1}{2} ||\nabla_t^{(0)} s||_{t,0}^2 + \left\langle T_{\frac{1}{4} c^i c^j R^L_{i,j}(x_0), p} s, s \right\rangle_{t,0} + O(t) ||s||_{t,0}^2. \tag{3.4.127}$$

Together with (3.4.35), this gives the first two estimates of (3.4.126).

By (3.3.69) and (3.4.117), (3.3.81) still holds, i.e.

$$\nabla_{t,e_i} - \nabla_{t,e_i}^{(0)} = \mathscr{O}_0\left(t^2\right), \tag{3.4.128}$$

where by $\mathscr{O}_0(t^\alpha)$ we mean an operator of order 0 which is a $O(t^\alpha)$. Thus, by (3.4.120) we again have

$$\mathscr{R}_t = \nabla_{t,e_i} \mathscr{O}_0(t) + \mathscr{O}_0(1), \tag{3.4.129}$$

hence the last estimate of (3.4.126) holds. □

Recall that the contour Γ in $\mathbb{C}$ has been defined in Figure 3.2 in Section 3.3.3.

Proposition 3.4.15. *There exist $C > 0$, $a, b \in \mathbb{N}$ such that for any $t > 0$ and any $\lambda \in \Gamma$, the resolvant $(\lambda - \mathscr{L}_t)^{-1}$ exists and*

$$\begin{aligned}
&\left\| (\lambda - \mathscr{L}_t)^{-1} \right\|_t^{0,0} \leq C(1 + |\lambda|^2)^a, \\
&\left\| (\lambda - \mathscr{L}_t)^{-1} \right\|_t^{-1,1} \leq C(1 + |\lambda|^2)^b.
\end{aligned} \tag{3.4.130}$$

Proof. Proposition 3.4.15 follows from Proposition 3.4.14 exactly as Proposition 3.3.16 follows from Proposition 3.3.15. □

Proposition 3.4.16. *Take $m \in \mathbb{N}^*$. Then there exists a contant $C_m > 0$ such that for any $t > 0$, $Q_1, \ldots, Q_m \in \left\{ \nabla_{t,e_i}^{(0)}, Z_i \right\}_{i=1}^{2n}$ and $s, s' \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0} X, \mathbb{E}_{p,x_0})$,*

$$\left| \left\langle [Q_1, [Q_2, \ldots [Q_m, \mathscr{L}_t] \ldots]] s, s' \right\rangle_{t,0} \right| \leq C_m ||s||_{t,1} ||s'||_{t,1}. \tag{3.4.131}$$

Proof. This Proposition is proved in the same way as Proposition 3.3.17. □

From Proposition 3.4.16, we can deduce the following result as done for Proposition 3.3.18.

Proposition 3.4.17. *For any $t > 0$, $\lambda \in \Gamma$ and $m \in \mathbb{N}$,*

$$(\lambda - \mathscr{L}_t)^{-1} (\boldsymbol{H}_t^m) \subset \boldsymbol{H}_t^{m+1}. \tag{3.4.132}$$

Moreover, for any $\alpha \in \mathbb{N}^{2n}$, there exist $K \in \mathbb{N}$ and $C_{\alpha,m} > 0$ such that for any $t > 0$, $\lambda \in \Gamma$ and $s \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0} X, \mathbb{E}_{p,x_0})$,

$$\left\| Z^\alpha (\lambda - \mathscr{L}_t)^{-1} s \right\|_{t,m+1} \leq C_{\alpha,m} (1 + |\lambda|^2)^K \sum_{\alpha' \leq \alpha} ||Z^{\alpha'} s||_{t,m}. \tag{3.4.133}$$

Let $e^{-\mathscr{L}_t}(Z,Z')$ be the smooth kernel of the operator $e^{-\mathscr{L}_t}$ with respect to $dv_{TX}(Z')$. Let $\mathrm{pr}_X\colon T_{\mathbb{R}}X\times_X T_{\mathbb{R}}X\to X$ be the projection from the fiberwise product $T_{\mathbb{R}}X\times_M T_{\mathbb{R}}X$ to M, then $e^{-\mathscr{L}_t}(\cdot,\cdot)$ is a section of $\mathrm{pr}_X^*\left(\mathrm{End}\left(\mathbb{E}_p\right)\right)$ over $T_{\mathbb{R}}X\times_M T_{\mathbb{R}}X$. Let $\nabla^{\mathrm{End}(\mathbb{E}_p)}$ be the connection on the bundle $\mathrm{End}(\mathbb{E}_p)$ over M induced by $\nabla^{\Lambda^\bullet(T^*_{\mathbb{R}}B)}$, $\nabla^{\Lambda^{0,\bullet},LC}$, ∇^{ξ} and ∇^{F_p}, and let $\nabla^{\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p)}$ be the induced connection on $\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p)$. Then $\nabla^{\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p)}$ and the operator norm on $\mathrm{End}(\mathbb{E}_p)$ induce naturally a $\mathscr{C}^m$-norm for the parameters $b_0\in B$ and $x_0\in X_{b_0}$.

Theorem 3.4.18. *For any $m,m'\in\mathbb{N}$, there is $C>0$ such that for any $t>0$, $Z,Z'\in T_{\mathbb{R},x_0}X$ with $|Z|,|Z'|\leq 1$,*

$$\sup_{|\alpha|,|\alpha'|\leq m}\left\|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}e^{-\mathscr{L}_t}(Z,Z')\right\|_{\mathscr{C}^{m'}(M,\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p))}\leq C, \tag{3.4.134}$$

where $|\cdot|_{\mathscr{C}^{m'}(M,\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p))}$ denotes the $\mathscr{C}^{m'}$ norm with respect to the parameters b_0 in a compact subset of B and $x_0\in X_{b_0}$.

Proof. For $m\in\mathbb{N}$ and $p\in\mathbb{N}^*$, let

$$\mathcal{Q}^m=\left\{\nabla^{(0)}_{t,e_{i_1}}\cdots\nabla^{(0)}_{t,e_{i_j}}\right\}_{j\leq m}. \tag{3.4.135}$$

As in the proof of Theorem 3.3.19 (see (3.3.102)-(3.3.104)), it follows from Proposition 3.4.17 that there exists $C_m>0$ such that for $p\in\mathbb{N}^*$ and $Q,Q'\in\mathcal{Q}^m$,

$$\left\|Qe^{-\mathscr{L}_t}Q'\right\|_t^{0,0}\leq C_m. \tag{3.4.136}$$

Here, we cannot conclude with a Sobolev inequality for a fixed Sobolev norm as in the proof of Theorem 3.3.19, because the space is changing. However, we will show a uniformity result in the Sobolev inequality for the “standard" Soboloev norm.

Lemma 3.4.19. *For every $d\in\mathbb{N}^*$, we endow $M_d(\mathbb{C})$ (the space of $d\times d$ matrices with coefficients in $\mathbb{C}$) with the operator norm $\|\cdot\|$. This induces a Sobolev norm on $\mathscr{C}^\infty(\mathbb{R}^N,M_d(\mathbb{C}))$ by*

$$||\varphi||_k^2=\sum_{\ell\leq k}\sum_{i_1,\ldots,i_\ell}\int_{\mathbb{R}^N}\|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_\ell}}\varphi(x)\|^2dx_1\ldots dx_N. \tag{3.4.137}$$

We denote the corresponding Sobolev space by $H^k(\mathbb{R}^N,M_d(\mathbb{C}))$. We also define the $\mathscr{C}^\ell$ norm $\|\cdot\|_{\mathscr{C}^\ell}$ on $\mathscr{C}^\infty_c(\mathbb{R}^N,M_d(\mathbb{C}))$ by

$$\|\varphi\|_{\mathscr{C}^\ell}=\sup_{\substack{j\leq\ell\\ i_1,\ldots,i_j}}\sup_{x\in\mathbb{R}^N}\|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_j}}\varphi(x)\| \tag{3.4.138}$$

Then for every $k,\ell\in\mathbb{N}$ such that $k-\ell>N/2$, there exists $C_{k,\ell,N}>0$ such that for every $d\in\mathbb{N}^*$ and $\varphi\in H^k(\mathbb{R}^N,M_d(\mathbb{C}))$,

$$\varphi \text{ is } \mathscr{C}^\ell \text{ and } \|\varphi\|_{\mathscr{C}^\ell}\leq C_{k,\ell,N}||\varphi||_k. \tag{3.4.139}$$

Proof. Suppose first that $\ell=0$. For $\varphi\in\mathscr{C}^\infty(\mathbb{R}^N,M_d(\mathbb{C}))$, we denote by $\widehat{\varphi}$ the Fourier transform of φ. Then by the Fourier inversion formula,

$$\varphi(x)=\frac{1}{(2\pi)^{N/2}}\int e^{ix\cdot\xi}\widehat{\varphi}(\xi)d\xi. \tag{3.4.140}$$

Thus, to show that φ is continuous, it suffices to prove that $\widehat{\varphi}(\xi)$ is in $L^1(\mathbb{R}^N, M_d(\mathbb{C}))$.

Set $\langle\xi\rangle = \left(1+|\xi|^2\right)^{1/2}$, and recall that

$$\varphi \in H^k(\mathbb{R}^N, M_d(\mathbb{C})) \iff \langle\xi\rangle^k \widehat{\varphi} \in L^2(\mathbb{R}^N, M_d(\mathbb{C})). \tag{3.4.141}$$

Moreover, there exists $c_{k,N} > 0$ independent of d such that for $\varphi \in H^k(\mathbb{R}^N, M_d(\mathbb{C}))$,

$$\frac{1}{c_{k,N}} \|\langle\xi\rangle^k \widehat{\varphi}\|_{L^2} \leq \|\varphi\|_k \leq c_{k,N} \|\langle\xi\rangle^k \widehat{\varphi}\|_{L^2}. \tag{3.4.142}$$

Now, we use Cauchy-Scwarz inequality:

$$\begin{aligned}\int |\widehat{\varphi}(\xi)| d\xi &\leq \int |\langle\xi\rangle^k \widehat{\varphi}(\xi)| \times |\langle\xi\rangle^{-k} \operatorname{Id}| d\xi \\ &\leq \|\langle\xi\rangle^k \widehat{\varphi}\|_{L^2} \int \langle\xi\rangle^{-2k} d\xi \\ &\leq C_{k,0,N} \|\varphi\|_k.\end{aligned} \tag{3.4.143}$$

The case $\ell \geq 1$ follows from the case $\ell = 0$ applied to the derivatives of φ. □

We can now finish the proof of Theorem 3.4.18, applying Lemma 3.4.19 to our situation. Let $m \in \mathbb{N}$, as $e^{-\mathscr{L}_t}(\cdot,\cdot) \in \mathscr{C}^\infty\left((T_{\mathbb{R},x_0}X)^2, \operatorname{End}(\mathbb{E}_p)\right)$, there is a $k \in \mathbb{N}$ and a constant $C > 0$ independent on p such that for $|\alpha|, |\alpha'| \leq m$ and $|Z|, |Z'| \leq 1$,

$$\left\| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} e^{-\mathscr{L}_t}(Z, Z') \right\| \leq C \left\| e^{-\mathscr{L}_t}(\cdot,\cdot)|_{B(0,1)^2} \right\|_k. \tag{3.4.144}$$

Now, by (3.4.117) and (3.4.123), for any $m \in \mathbb{N}$ there exists $C'_m > 0$ independent on t such that for $\varphi \in \mathscr{C}^\infty((T_{\mathbb{R},x_0}X)^2, \operatorname{End}(\mathbb{E}_p))$ with support in $B^{T_{\mathbb{R},x_0}}(0,1)^2$,

$$\frac{1}{C'_m} \|\varphi\|_{t,m} \leq \|\varphi\|_m \leq C'_m \|\varphi\|_{t,m}. \tag{3.4.145}$$

With (3.4.136), (3.4.144) and (3.4.145), we see that (3.3.99) holds when $m' = 0$.

For $m' \geq 1$, we use the same argument as in Theorem 3.3.19 (equations (3.3.108) and (3.3.109)). □

Theorem 3.4.20. *There are constants $C > 0$ and $M \in \mathbb{N}^*$ such that for $t > 0$,*

$$\left\| ((\lambda - \mathscr{L}_t)^{-1} - (\lambda - \underline{\mathscr{L}}_t)^{-1}) s \right\|_{t,0} \leq Ct(1+|\lambda|^2)^M \sum_{|\alpha| \leq 3} ||Z^\alpha s||_{t,0}. \tag{3.4.146}$$

Proof. For $m = 0$, we denote in this proof $\|\cdot\|_{\underline{t},0} = \|\cdot\|_{t,0}$.

From (3.4.117) and (3.4.123), for $p, m \in \mathbb{N}^*$ we find

$$\|s\|_{t,m} \leq C \sum_{|\alpha| \leq m} \|Z^\alpha s\|_{\underline{t},m}. \tag{3.4.147}$$

Moreover, for s, s' with compact support, using Theorem 3.4.1 and a Taylor expansion of (3.4.120), we find

$$\left| \langle (\mathscr{L}_t - \underline{\mathscr{L}}_t) s, s' \rangle_{t,0} \right| \leq Ct \|s'\|_{t,1} \sum_{|\alpha| \leq 3} \|Z^\alpha s\|_{\underline{t},1}. \tag{3.4.148}$$

Thus,

$$\|(\mathscr{L}_t - \underline{\mathscr{L}}_t)s\|_{t,-1} \leq C \sum_{|\alpha|\leq 3} \|Z^\alpha s\|_{\underline{t},1}. \tag{3.4.149}$$

We have

$$(\lambda - \mathscr{L}_t)^{-1} - (\lambda - \underline{\mathscr{L}}_t)^{-1} = (\lambda - \mathscr{L}_t)^{-1}(\mathscr{L}_t - \underline{\mathscr{L}}_t)(\lambda - \underline{\mathscr{L}}_t)^{-1}. \tag{3.4.150}$$

Moreover, Propositions 3.4.15, 3.4.16 and 3.4.17 still hold for the operator $\underline{\mathscr{L}}_t$, the norms $\|\cdot\|_{\underline{t},m}$ and the family of test operators for commutators $\left\{\underline{\nabla}_{t,e_i}, Z_i\right\}_{i=1}^{2n}$. Thus, Proposition 3.4.17, (3.4.149) and (3.4.150) yields to (3.4.146). □

Proof of Theorem 3.4.13. By Theorems 3.4.18 and 3.4.20, we can prove Theorem 3.4.13 exactly as Theorem 3.3.21. □

Define

$$\underline{\mathscr{L}}_{t,u} = u\psi_{1/\sqrt{u}}\underline{\mathscr{L}_t}\psi_{\sqrt{u}}. \tag{3.4.151}$$

Whereas in Section 3.3.3 we could use a closed formula for the heat kernel of $\mathscr{L}_{0,u}$ to derive Theorem 3.3.11 from Theorem 3.3.21, here we cannot compute $e^{-\underline{\mathscr{L}}_{t,u}}(0,0)$ exactly to get the asymptotic of $\psi_{1/\sqrt{p}}\exp(-B^2_{p,u/p})(x_0,x_0)$. The difficulty is that here, the harmonic oscillator $\underline{\mathscr{L}}_t$ has its coefficients in the non-commutative algebra $\mathcal{T}_{x_0}$. However, by (3.4.43), the coefficients of $\underline{\mathscr{L}}_t$ tends to commute increasingly, so we can expect to have at least a equivalent of $e^{-\underline{\mathscr{L}}_{t,u}}(0,0)$.

For $y \in Y_{x_0}$, we define the operator $\mathscr{H}_{x_0}(y)$ acting on the space

$$\mathscr{C}^\infty\left(T_{\mathbb{R},x_0}X, \Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \left(\Lambda^{0,\bullet}(T^*X)\otimes\xi\right)_{x_0}\right) \tag{3.4.152}$$

by

$$\mathscr{H}_{x_0}(y) = -\frac{1}{2}\sum_i \left(\nabla_{e_i} + \frac{1}{2}R^L_{(x_0,y)}(Z^H, e_i^H)\right)^2 \\ + \frac{1}{4}c^ic^jR^L_{i,j}(x_0,y) + \frac{1}{\sqrt{2}}c^if^\alpha R^L_{i,\alpha}(x_0,y) + \frac{f^\alpha f^\beta}{2}R^L_{\alpha,\beta}(x_0,y). \tag{3.4.153}$$

Set also

$$\mathscr{H}_{x_0,u}(y) = u\psi_{1/\sqrt{u}}\mathscr{H}_{x_0}(y)\psi_{\sqrt{u}}. \tag{3.4.154}$$

Then $y \mapsto \mathscr{H}_{x_0}(y)$ is a smooth function from Y_{x_0} to the space of differential operators acting on the space given in (3.4.152). As a consequence, the family $\{P_{p,x_0}\mathscr{H}_{x_0}(y)P_{p,x_0}\}_p$ is a family of differential operators that belongs to the algebra $\mathcal{B}_{x_0}$. Now, as ∇_{e_i} and P_{p,x_0} commute, it is easy to see that for any $p \in \mathbb{N}^*$,

$$\underline{\mathscr{L}}_t = P_{p,x_0}\mathscr{H}_{x_0}(\cdot)P_{p,x_0}. \tag{3.4.155}$$

We denote by $e^{-\underline{\mathscr{L}}_t}(Z,Z')$ and $e^{-\mathscr{H}_{x_0}(y)}(Z,Z')$ the smooth kernels of the operators $e^{-\underline{\mathscr{L}}_t}$ and $e^{-\mathscr{H}_{x_0}(y)}$ with respect to $dv_{TX}(Z')$. Then for $Z, Z' \in T_{\mathbb{R},x_0}X$,

$$\left\{y \mapsto e^{-\mathscr{H}_{x_0}(y)}(Z,Z')\right\} \in \mathscr{C}^\infty\Big(Y_{x_0}, \Lambda^\bullet(T^*_{b_0}B)\otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*_{x_0}X)\otimes\xi_{x_0})\Big). \tag{3.4.156}$$

Theorem 3.4.21. *For $u>0$ fixed and for all $Z, Z' \in T_{\mathbb{R},x_0}X$ we have as $t \to 0$*

$$e^{-u\underline{\mathscr{L}}_t}(Z,Z') = T_{e^{-u\mathscr{H}_{x_0}(\cdot)}(Z,Z'),p} + o(1), \tag{3.4.157}$$

where $o(1)$ denotes a term converging to 0 for the operator norm.

Proof. For $\lambda \in \Gamma$ (see Figure 3.2), both $\lambda - P_{p,x_0}\mathscr{H}_{x_0}(y)P_{p,x_0}$ and $\lambda - \mathscr{H}_{x_0}(y)$ are invertible, so we can use a contour integral to get

$$\begin{aligned} e^{-uP_{p,x_0}\mathscr{H}_{x_0}(y)P_{p,x_0}} &- P_{p,x_0}e^{-u\mathscr{H}_{x_0}(\cdot)}P_{p,x_0} = \\ &\frac{1}{2i\pi}\int_\Gamma e^{-u\lambda}\Big[(\lambda - P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0})^{-1} - P_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0}\Big]\,d\lambda. \end{aligned} \tag{3.4.158}$$

Moreover, we have

$$\begin{aligned} (\lambda - P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0})^{-1} &- P_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0} \\ &= (\lambda - P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0})^{-1}(P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0} - \mathscr{H}_{x_0})P_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0} \\ &= (\lambda - P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0})^{-1}P_{p,x_0}\mathscr{H}_{x_0}P^\perp_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0}, \end{aligned} \tag{3.4.159}$$

where $P^\perp_{p,x_0} = 1 - P_{p,x_0}$.

As said earlier, Propositions 3.4.15, 3.4.16 and 3.4.17 (and thus Theorem 3.4.18) still hold for the operator $\underline{\mathscr{L}}_t$, the norms $\|\cdot\|_{\underline{t},m}$ and the family of test operators for commutators $\left\{\underline{\nabla}_{t,e_i}, Z_i\right\}_{i=1}^{2n}$. In the sequel, we will use these results without further notice.

By (3.4.130), we know that there are constants $C>0$ and $a \in \mathbb{N}$ such that for $\lambda \in \Gamma$,

$$\left\|(\lambda - P_{p,x_0}\mathscr{H}_{x_0}P_{p,x_0})^{-1}\right\|_t^{0,0} \le C(1+|\lambda|^2)^a, \tag{3.4.160}$$

so by (3.4.159), to estimate the norm of (3.4.158), we need to estimate

$$\left\|P_{p,x_0}\mathscr{H}_{x_0}P^\perp_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0}\right\|_t^{0,0}. \tag{3.4.161}$$

If $\mathscr{H}_{x_0}$ were just a function, this could be done using the first equation of (3.4.43). The idea is thus to extend (3.4.43) to the present case.

Remark 3.4.22. As ∇_{e_i} commutes with P_{p,x_0} and by (3.4.153), to estimate the norm in (3.4.161) we only have to compare

$$P_{p,x_0}fP_{p,x_0}(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0} \qquad \text{and} \qquad P_{p,x_0}f(\lambda - \mathscr{H}_{x_0})^{-1}P_{p,x_0} \tag{3.4.162}$$

for a function $f \in \mathscr{C}^\infty(T_{\mathbb{R},x_0}X \times Y_{x_0}, \mathbb{C})$.

Note that $y \mapsto (\lambda - \mathscr{H}_{x_0}(y))^{-1}$ is a smooth function on Y_{x_0} with values in the algebra of bounded operator acting on the Hilbert space

$$L^2(T_{\mathbb{R},x_0}X, \Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \Big(\Lambda^{0,\bullet}(T^*X)\otimes\xi\Big)_{x_0}). \tag{3.4.163}$$

Thus, we can apply Theorem 3.4.5 to our situation, that is

$$\begin{aligned} &\mathcal{H} = L^2(T_{\mathbb{R},x_0}X, \Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \Big(\Lambda^{0,\bullet}(T^*X)\otimes\xi\Big)_{x_0});\\ &A(y) = f(\cdot,y);\\ &B(y) = (\lambda - \mathscr{H}_{x_0}(y))^{-1}. \end{aligned} \tag{3.4.164}$$

We then get

$$P_{p,x_0} f P_{p,x_0} (\lambda - \mathscr{H}_{x_0})^{-1} P_{p,x_0} - P_{p,x_0} f (\lambda - \mathscr{H}_{x_0})^{-1} P_{p,x_0} = O(p^{-1}). \tag{3.4.165}$$

Here, the term $O(p^{-1})$ depends of course on λ. To get the expansion (3.4.66), we used the Taylor expansion of A. Thus, in (3.4.66), we can bound the error term $O(p^{-\frac{k+1}{2}})$ using the derivatives of A of order less than $k+1$. Applying this argument to $(\lambda - \mathscr{H}_{x_0})^{-1}$ and using Proposition 3.4.15, we find that there exists $M \in \mathbb{N}^*$ such that

$$\left\| P_{p,x_0} f P_{p,x_0} (\lambda - \mathscr{H}_{x_0})^{-1} P_{p,x_0} - P_{p,x_0} f (\lambda - \mathscr{H}_{x_0})^{-1} P_{p,x_0} \right\|_t^{0,0} \leq C p^{-1} (1 + |\lambda|^2)^M. \tag{3.4.166}$$

Hence, by Remark 3.4.22,

$$\left\| P_{p,x_0} \mathscr{H}_{x_0} P^{\perp}_{p,x_0} (\lambda - \mathscr{H}_{x_0})^{-1} P_{p,x_0} \right\|_t^{0,0} \leq C p^{-1} (1 + |\lambda|^2)^M. \tag{3.4.167}$$

With (3.4.158), (3.4.159), (3.4.160) and (3.4.167) we infer that

$$\left\| e^{-P_{p,x_0} \mathscr{H}_{x_0}(y) P_{p,x_0}} - P_{p,x_0} e^{-\mathscr{H}_{x_0}(\cdot)} P_{p,x_0} \right\|_t^{0,0} \leq C p^{-1}. \tag{3.4.168}$$

Note that $P_{p,x_0} e^{-\mathscr{H}_{x_0}(\cdot)} P_{p,x_0}$ satisfies a estimate analogous to (3.4.134). Indeed, we have

$$P_{p,x_0} e^{-\mathscr{H}_{x_0}(\cdot)} P_{p,x_0}(Z, Z') = P_{p,x_0} e^{-\mathscr{H}_{x_0}(\cdot)}(Z, Z') P_{p,x_0}, \tag{3.4.169}$$

and we can apply (3.3.99) to $\mathscr{H}_{x_0}(y)$ (which correspond for y fixed to $\mathscr{L}_{\infty,u}$ in Section 3.3.2) and (3.4.35) to conclude. Thus, by (3.4.134) applied to $e^{-P_{p,x_0} \mathscr{H}_{x_0}(y) P_{p,x_0}}$ and $P_{p,x_0} e^{-\mathscr{H}_{x_0}(\cdot)} P_{p,x_0}$, and by (3.4.168), we can apply the method of Theorem 3.3.21 to complete the proof of Theorem 3.4.21. □

Using the analogue of Lemma 3.4.11, Theorems 3.4.13 and 3.4.21, and (3.3.122) we get that

$$\psi_{1/\sqrt{p}} e^{-B^2_{p,u/p}}(x_0, x_0) = p^{n_X} T_{e^{-\mathscr{H}_{x_0,u}(\cdot)}(0,0),p} + o(p^{n_X}) \tag{3.4.170}$$

for the operator norm and the operator norm of the derivatives. Now, comparing the definitions of $\mathscr{H}_{x_0,u}$ in (3.4.153) and (3.4.154) and of $\mathscr{L}_{0,u}$ in (3.3.65), and using (3.3.125), we find that

$$T_{e^{-\mathscr{H}_{x_0,u}(\cdot)}(0,0),p} = (2\pi)^{-n_X} P_{p,x_0} \exp(-\Omega_{u,(x_0,\cdot)}) \frac{\det(\dot{R}^{X,L}_{(x_0,\cdot)})}{\det\left(1 - \exp(-u \dot{R}^{X,L}_{(x_0,\cdot)})\right)} \otimes \mathrm{Id}_\xi \, P_{p,x_0}, \tag{3.4.171}$$

where $\dot{R}^{X,L}$ is define in (3.4.3) and

$$\Omega_u = u R^L(w_k^H, \overline{w}_\ell^H) \overline{w}^\ell \wedge i_{\overline{w}_k} + \sqrt{\frac{u}{2}} c(e_i) f^\alpha R^L(e_i^H, f_\alpha^H) + \frac{f^\alpha f^\beta}{2} R^L(f_\alpha^H, f_\beta^H). \tag{3.4.172}$$

Finally, we have proved that for any $k \in \mathbb{N}$, as $p \to +\infty$, uniformly as u varies in a compact subset of $\mathbb{R}_+^*$, we have the following asymptotic for for the operator norm on $\mathrm{End}(\mathbb{E}_p)$ and the operator norm of the derivatives up to order k:

$$\begin{aligned} &\psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})(x_0, x_0) \\ &\quad = \frac{p^{n_X}}{(2\pi)^{n_X}} P_{p,x_0} e^{-\Omega_{u,(x_0,\cdot)}} \frac{\det(\dot{R}^{X,L}_{(x_0,\cdot)})}{\det\left(1 - \exp(-u \dot{R}^{X,L}_{(x_0,\cdot)})\right)} \otimes \mathrm{Id}_{\xi_{x_0}} P_{p,x_0} + o(p^{n_X}). \end{aligned} \tag{3.4.173}$$

Theorem 3.1.9 is proved.

3.4.7 Asymptotic of the torsion forms.

The method is the same as in Section 3.3.4. Let $b_0 \in B$, we denote X_{b_0} and Z_{b_0} simply by X and Z. Let $d = \dim M$.

Definition 3.4.23. Let $\Lambda \in \mathscr{C}^\infty\Big(Z, \pi_1^*\mathrm{End}\big(\Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \Lambda^{0,\bullet}(T^*X)\big)\Big)$ be defined by

$$\Lambda_u(z) = e^{-\mathscr{H}_u(z)}(0,0) = (2\pi)^{-n_X}\exp(-\Omega_{u,z})\frac{\det(\dot{R}^{X,L}_z)}{\det\big(\mathrm{Id} - \exp(-u\dot{R}^{X,L}_z)\big)}, \tag{3.4.174}$$

and let $\mathrm{R}_u \in \mathscr{C}^\infty(Z,\mathbb{C})$ be defined by

$$\mathrm{R}_u(z) = \mathrm{Tr}_s\left[N_u\Lambda_u(z)\right]. \tag{3.4.175}$$

Let $A_j \in \mathscr{C}^\infty\Big(Z, \pi_1^*\mathrm{End}\big(\Lambda^\bullet(T^*_{\mathbb{R},b_0}B) \otimes \Lambda^{0,\bullet}(T^*X)\big)\Big)$ be such that as $u \to 0$

$$\Lambda_u(z) = \sum_{j=-d}^{k} A_j(z)u^j + O(u^{k+1}), \tag{3.4.176}$$

and here again we set $A_{-d-1} = 0$.

Theorem 3.4.24. *There exist $\{A_{p,j}\} \in \mathscr{C}^\infty(X, \mathrm{End}(\mathbb{E}_p))$ such that for any $k, \ell \in \mathbb{N}$, there exist $C > 0$ such that for any $u \in]0,1]$ and $p \geq 1$,*

$$\left\| p^{-n_X}\psi_{1/\sqrt{p}}\exp\left(-B^2_{p,u/p}\right)(x,x) - \sum_{j=-d}^{k} A_{p,j}(x)u^j \right\|_{\mathscr{C}^\ell(M)} \leq Cu^{k+1}. \tag{3.4.177}$$

Moreover, as $p \to +\infty$, we have for any $j \geq -d$

$$A_{p,j}(x) = P_{p,x}A_j(x,\cdot)\otimes \mathrm{Id}_{\xi_x} P_{p,x} + o(1), \tag{3.4.178}$$

for the operator norm on $\mathrm{End}(\mathbb{E}_p)$ and the operator norm of the derivatives up to order ℓ.

Theorem 3.4.24 will be proved in Section 3.4.8.
For $j \geq -d-1$, set

$$\tilde{A}_j(z) = \mathrm{Tr}_s\left[N_V A_j(z) + i\omega^H A_{j+1}(z)\right]. \tag{3.4.179}$$

Then by (3.2.39), (3.4.175) and (3.4.176), we have

$$\mathrm{R}_u(z) = \sum_{j=-n-1}^{k} \tilde{A}_j(z)u^j + O(u^{k+1}). \tag{3.4.180}$$

Set also

$$\begin{aligned} B_{p,j} &= \int_Z \mathrm{Tr}_s\left[N_V A_{p,j}(z) + i\omega^H A_{p,j+1}(z)\right]\frac{\Theta^{Y,m}}{m!}dv_X, \\ B_j &= \int_Z \tilde{A}_j(z)\frac{\Theta^{Y,m}}{m!}dv_X. \end{aligned} \tag{3.4.181}$$

Recall that $r = \dim Z = n + m$.

Corollary 3.4.25. *For any $k, l \in \mathbb{N}$, there exists $C > 0$ such that for any $u \in]0,1]$ and $p \geq 1$,*

$$\left| p^{-n_Z} \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp\left(-B^2_{p,u/p}\right)\right] - \sum_{j=-d-1}^{k} B_{p,j} u^j \right| \leq C u^{k+1}. \tag{3.4.182}$$

Moreover, as $p \to +\infty$, we have for any $j \geq -d$

$$B_{p,j} = \operatorname{rk}(\xi)\operatorname{rk}(\eta) B_j + O\left(\frac{1}{\sqrt{p}}\right). \tag{3.4.183}$$

Proof. This is a consequence of (3.4.36) and Theorem 3.4.24. □

Theorem 3.4.26. *There exists $C > 0$ such that for $u \geq 1$ and $p \geq 1$,*

$$\left| p^{-n_Z} \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp\left(-B^2_{p,u/p}\right)\right]\right| \leq \frac{C}{\sqrt{u}}. \tag{3.4.184}$$

Theorem 3.4.26 will be proved in Section 3.4.9.

Let p_0 be such that for all $p \geq p_0$, the direct images $R^\bullet \pi_{1,*}(\eta \otimes L^p)$ is locally free, $R^i \pi_{1,*}(\eta \otimes L^p) = 0$ for $i > 0$ and the direct images $R^\bullet \pi_{2,*}(\xi \otimes F_p)$ and $R^\bullet \pi_{3,*}(\pi_1^* \xi \otimes \eta \otimes L^p)$ are locally free.

As in Section 3.3.4, we define for $p \geq p_0$

$$\tilde{\zeta}_p(s) = -\frac{p^{-n_Z}}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \psi_{1/\sqrt{p}} \Phi \left\{ \operatorname{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p})\right]\right\} du. \tag{3.4.185}$$

Then if ζ_p denotes the zeta function (3.2.48) associated with $B_{p,u}$, we have

$$\begin{aligned} p^{-n_Z} \psi_{1/\sqrt{p}} \zeta_p(s) &= p^{-s} \tilde{\zeta}_p(s), \\ p^{-n_Z} \psi_{1/\sqrt{p}} \zeta'_p(0) &= \log(p) B_{p,0} + \tilde{\zeta}'_p(0). \end{aligned} \tag{3.4.186}$$

Let

$$\tilde{\zeta}(s) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} \int_Z \mathrm{R}_u(z) dv_Z(z) u^{s-1} du. \tag{3.4.187}$$

As in Section 3.3.4, by (3.4.36) and Theorem 3.1.9, and by dominated convergence (justified by Corollary 3.4.25 and Theorem 3.4.26) we find that

$$\tilde{\zeta}'_p(0) \xrightarrow[p \to +\infty]{} \operatorname{rk}(\xi)\operatorname{rk}(\eta) \Phi \tilde{\zeta}'(0). \tag{3.4.188}$$

Let $T^{H'}_B N \subset T^H_M N$ be the space obtained by lifting in TN the subspace $T^H_B M$ of TM. In particular, $T^{H'}_B N$ is orthogonal to TY. Let $\{f'_\alpha\}$ be an orthonormal basis of $T^{H'}_{B,\mathbb{R}} N$ with dual basis $\{f'^\alpha\}$. Set

$$\mathscr{F}^H = \exp\left(-f'^\alpha f'^\beta R^L(f'_\alpha, f'_\beta)\right). \tag{3.4.189}$$

Repeating the computations done in the proof of Theorem 3.3.26 which yield to (3.3.169) and (3.3.173), we find here again that

$$\begin{aligned} &\tilde{A}_j = 0 \text{ for } j \leq -2, \\ &\mathrm{R}_u - \frac{\tilde{A}_{-1}}{u} - \tilde{A}_0 = \left\{ \mathrm{R}^{\{*\}}_u - \frac{\tilde{A}^{\{*\}}_{-1}}{u} - \tilde{A}^{\{*\}}_0 \right\} \mathscr{F}^H. \end{aligned} \tag{3.4.190}$$

Thus, we have

$$\tilde{\zeta}'(0) = \frac{1}{2}\int_Z \det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\log\left[\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\right]\mathscr{F}^H\frac{\Theta^{Y,m}}{m!}dv_X. \tag{3.4.191}$$

Moreover, by (3.4.189), we know that

$$\begin{aligned}&\det\left(\frac{\dot{R}^{X,L}}{2\pi}\right)\frac{\Theta^{Y,m}}{m!}dv_X = \frac{\Theta^{Z,r}}{r!},\\ &\Phi\mathscr{F}^H e^{\Theta^Z} = e^{\Theta^N}.\end{aligned} \tag{3.4.192}$$

Thus, by Corollary 3.4.25, (3.4.186), (3.4.188), (3.4.191) and as in (3.3.180), we have as $p \to +\infty$

$$\begin{aligned}\psi_{1/\sqrt{p}}\zeta_p'(0) &= \log(p)p^{n_Z}B_0 + p^{n_Z}\Phi\tilde{\zeta}'(0) + o(p^{n_Z})\\ &= \frac{\mathrm{rk}(\xi)\mathrm{rk}(\eta)}{2}\int_Z \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right]e^{\Theta^N+(p-1)\Theta^Z} + o(p^{n_Z}).\end{aligned} \tag{3.4.193}$$

Theorem 3.1.7 is proved.

3.4.8 Proof of Theorem 3.4.24

First, we would like to point out that we cannot use the method of proof of Theorem 3.3.23 used in Section 3.3.5 to prove Theorem 3.4.24. Indeed, the point was to see t as a parameter, in the same way as x_0, and to use the fact that the development of the heat kernel on a compact space acting on a *fixed* bundle is smooth in the parameters. However, here we cannot fixe the bundle, so we have to reprove directly the uniform development of the heat kernel. The techniques in this section are inspired by [46, Sect. 4.1].

Let ∇ be the usual derivation and let $\Delta^{T_{\mathbb{R},x_0}X}$ be the usual Bochner Laplacian on $T_{\mathbb{R},x_0}X$. Recall that ρ is defined in (3.3.54), and define

$$\mathscr{L}_{2,t} = \rho(|Z|/\varepsilon)\mathscr{L}_t + \left(1-\rho(|Z|/\varepsilon)\right)\Delta^{T_{\mathbb{R},x_0}X}. \tag{3.4.194}$$

Then using (3.3.181) as in Proposition 3.4.8 and Lemma 3.4.11, we find

$$\left\|e^{-u\mathscr{L}_t}(0,0) - e^{-u\mathscr{L}_{2,t}}(0,0)\right\|_{\mathscr{C}^m(M)} \leq C\exp(-\frac{\varepsilon^2 p}{32u}). \tag{3.4.195}$$

For $v = \sqrt{u}$, set (with S_v in (3.3.65))

$$\begin{aligned}\mathscr{L}_{3,t}^v &= v^2 S_v^{-1}\mathscr{L}_{2,t}S_v,\\ \mathscr{L}_{3,t}^0 &= \Delta^{T_{\mathbb{R},x_0}X}.\end{aligned} \tag{3.4.196}$$

Then as in (3.3.120), we have

$$e^{-u\mathscr{L}_{2,t}}(0,0) = u^{-n}e^{-\mathscr{L}_{3,t}^v}(0,0). \tag{3.4.197}$$

We will use the usual Sobolev norm $\|\cdot\|_k$ (see (3.4.137)) on $\mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbb{E}_{p,x_0})$.

Using the fact that uniformly in t we have

$$\mathscr{L}_{3,t}^v = \Delta^{T_{\mathbb{R},x_0}X} + O(v), \tag{3.4.198}$$

we get the following analogues of Propositions 3.4.14 to 3.4.17, replacing $\nabla_t^{(0)}$, $\mathscr{L}_t$ and $\|\cdot\|_{t,k}$ by ∇, $\mathscr{L}_{3,t}^v$ and $\|\cdot\|_k$.

Set

$$\mathscr{R}_{3,t}^v = \mathscr{L}_{3,t}^v - \mathscr{L}_{3,t}^{v,(0)}. \tag{3.4.199}$$

Proposition 3.4.27. *There exist constants $C_1, C_2, C_3 > 0$ such that for any $t > 0$, $v \geq 0$ and $s, s' \in \mathscr{C}^\infty(X_0, \mathbb{E}_{p,x_0})$,*

$$\begin{aligned} &\langle \mathscr{L}_{3,t}^{v,(0)} s, s \ \rangle_0 \geq C_1 ||s||_1^2 - C_2 ||s||_0^2, \\ &\left| \langle \mathscr{L}_{3,t}^{v,(0)} s, s' \ \rangle_0 \right| \leq C_3 ||s||_1 ||s'||_1, \\ &\left\| \mathscr{R}_{3,t}^v s \right\|_0 \leq C_4 ||s||_1. \end{aligned} \tag{3.4.200}$$

Proposition 3.4.28. *There exist $C > 0$, $a, b \in \mathbb{N}$ such that for any $t > 0$, $v \geq 0$ and $\lambda \in \Gamma$, the resolvant $(\lambda - \mathscr{L}_t)^{-1}$ exists and*

$$\begin{aligned} &\left\| \left(\lambda - \mathscr{L}_{3,t}^v \right)^{-1} \right\|^{0,0} \leq C(1 + |\lambda|^2)^a, \\ &\left\| \left(\lambda - \mathscr{L}_{3,t}^v \right)^{-1} \right\|^{-1,1} \leq C(1 + |\lambda|^2)^b. \end{aligned} \tag{3.4.201}$$

Proposition 3.4.29. *Take $m \in \mathbb{N}^*$. Then there exists a contant $C_m > 0$ such that for any $t > 0$, $v \geq 0$, $Q_1, \ldots, Q_m \in \{\nabla_{e_i}, Z_i\}_{i=1}^{2n}$ and $s, s' \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0} X, \mathbb{E}_{p,x_0})$,*

$$\left| \left\langle [Q_1, [Q_2, \ldots [Q_m, \mathscr{L}_{3,t}^v] \ldots]] s, s' \right\rangle_0 \right| \leq C_m ||s||_1 ||s'||_1. \tag{3.4.202}$$

Proposition 3.4.30. *For any $t > 0$, $\lambda \in \Gamma$ and $m \in \mathbb{N}$,*

$$(\lambda - \mathscr{L}_{3,t}^v)^{-1} (\boldsymbol{H}^m) \subset \boldsymbol{H}^{m+1}. \tag{3.4.203}$$

Moreover, for any $\alpha \in \mathbb{N}^{2n}$, there exist $K \in \mathbb{N}$ and $C_{\alpha,m} > 0$ such that for any $t > 0$, $v \geq 0$, $\lambda \in \Gamma$ and $s \in \mathscr{C}_0^\infty(T_{\mathbb{R},x_0} X, \mathbb{E}_{p,x_0})$,

$$\left\| Z^\alpha (\lambda - \mathscr{L}_{3,t}^v)^{-1} s \right\|_{m+1} \leq C_{\alpha,m} (1 + |\lambda|^2)^K \sum_{\alpha' \leq \alpha} ||Z^{\alpha'} s||_m. \tag{3.4.204}$$

For $k, q \in \mathbb{N}^*$, set

$$I_{k,r} = \Big\{ (\mathbf{k}, \mathbf{r}) = (k_i, r_i) \in (\mathbb{N}^*)^{j+1} \times (\mathbb{N}^*)^j \ : \ \sum_{i=0}^j k_i = k + j \, , \, \sum_{i=1}^j r_i = r \Big\}. \tag{3.4.205}$$

For $(\mathbf{k}, \mathbf{r}) \in I_{k,r}$, $\lambda \in \Gamma$ (see Figure 3.2 in Section 3.3.3), $t > 0$ and $v \geq 0$ set

$$A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t, v) = (\lambda - \mathscr{L}_{3,t}^v)^{-k_0} \frac{\partial^{r_1} \mathscr{L}_{3,t}^v}{\partial v^{r_1}} (\lambda - \mathscr{L}_{3,t}^v)^{-k_1} \cdots \frac{\partial^{r_j} \mathscr{L}_{3,t}^v}{\partial v^{r_j}} (\lambda - \mathscr{L}_{3,t}^v)^{-k_j}. \tag{3.4.206}$$

Then there exist $a_{\mathbf{r}}^{\mathbf{k}} \in \mathbb{R}$ such that

$$\frac{\partial^r}{\partial v^r} (\lambda - \mathscr{L}_{3,t}^v)^{-k} = \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,q}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t, v). \tag{3.4.207}$$

For $\ell \in \mathbb{N}$, let $\mathcal{Q}^\ell$ be the set of operators

$$\mathcal{Q}^\ell = \{ \nabla_{e_{i_1}} \ldots \nabla_{e_{i_j}} \}_{j \leq \ell}. \tag{3.4.208}$$

Theorem 3.4.31. *For any $\ell \in \mathbb{N}$, $k > 2(\ell + r + 1)$ and $(\mathbf{k}, \mathbf{r}) \in I_{k,r}$, there are $C_m > 0$ and $N \in \mathbb{N}$ such that for any $\lambda \in \Gamma$, $t > 0$, $v \geq 0$ and $Q, Q' \in \mathcal{Q}^\ell$,*

$$\left\| Q A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, v) Q' \right\|_0 \leq C(1 + |\lambda|)^N \sum_{|\beta| \leq 2r} \| Z^\beta s \|_0. \tag{3.4.209}$$

Proof. First, note that as in the proof of Theorem 3.3.19 (see (3.3.102)-(3.3.104)), Proposition 3.4.30 leads to

$$\left\| Q(\lambda - \mathscr{L}^v_{3,t})^{-m} \right\|^{0,0} \leq C(1 + |\lambda|)^N, \quad \left\| (\lambda - \mathscr{L}^v_{3,t})^{-m} Q' \right\|^{0,0} \leq C(1 + |\lambda|)^N. \tag{3.4.210}$$

With this estimate and Proposition 3.4.28, we get (3.4.209) for $r = 0$.

Assume now $r > 0$. By (3.4.117), (3.4.120), (3.4.194), (3.4.196) and Theorem 3.4.1, we know that $\frac{\partial^r}{\partial v^r} \mathscr{L}^v_{3,t}$ is a combination of

$$\begin{aligned}
&\Big(\frac{\partial^{r_1}}{\partial v^{r_1}} a_{ij}(t, vZ) \Big) \Big(\frac{\partial^{r_2}}{\partial v^{r_2}} \nabla^v_{3,t,e_i} \Big) \Big(\frac{\partial^{r_3}}{\partial v^{r_3}} \nabla^v_{3,t,e_j} \Big), && \frac{\partial^{r_1}}{\partial v^{r_1}} b(t, vZ), \\
&\Big(\frac{\partial^{r_1}}{\partial v^{r_1}} c_i(t, vZ) \Big) \Big(\frac{\partial^{r_2}}{\partial v^{r_2}} \nabla^v_{3,t,e_i} \Big), && \Big(\frac{\partial^{r_1}}{\partial v^{r_1}} d(t, vZ) \Big) \Delta^{T_{\mathbb{R},x_0} X},
\end{aligned} \tag{3.4.211}$$

where a_{ij}, b, c_i and d are of the form $f(Z) g(tZ)$ with $f(Z)$ and $g(Z)$ and their derivatives in Z uniformly bounded for $Z \in \mathbb{R}^{2n}$ (recall that for Toeplitz operators, we take the operator norm).

Now, if e is one of a_{ij}, b, c_i or d, then for $r_1 \geq 1$, $\frac{\partial^{r_1}}{\partial v^{r_1}} e(t, vZ)$ (resp. $\frac{\partial^{r_1}}{\partial v^{r_1}} \nabla^v_{3,t,e_i}$) is a function of the type $f(vZ) g(tvZ) Z^\beta$ with $|\beta| \leq r_1$ (resp. $r_1 + 1$) and $f(Z)$ and $g(Z)$ and their derivatives in Z uniformly bounded for $Z \in \mathbb{R}^{2n}$.

Let $\mathscr{F}'_{t,v}$ be the family of operators of the form

$$\mathscr{F}'_{t,v} = \{ [f_{j_1} Q_{j_1}, [f_{j_2} Q_{j_2}, \ldots [f_{j_m} Q_{j_m}, \mathscr{L}^v_{3,t}] \ldots]] \}, \tag{3.4.212}$$

where f_{j_i} is smooth and bounded (with its derivatives) and $Q_{j_i} \in \{ \nabla_{e_l}, Z_l \}_{l=1}^{2n}$.

We will now deal with the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, v) Q'$. First, we move all the terms Z^β in the terms $f(vZ) g(tvZ) Z^\beta$ (defined above) to the right-hand side of this operator. To do so, we use the same commutator trick as in the proof of Theorem 3.3.18, that is we perform the commutations once at a time with each Z_i (and not directly with Z^β, $|\beta| > 1$). Then we obtain that $A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, v) Q'$ is of the form $\sum_{|\beta| \leq 2r} L^v_{t,\beta} Q''_\beta Z^\beta$ where Q''_β is obtained from Q' and its commutation with Z^β. Next, we move all the terms ∇^v_{3,t,e_i} in $\frac{\partial^r}{\partial v^r} \mathscr{L}^v_{3,t}$ to the right-hand side of the operators $L^v_{t,\beta}$. Then as in the proof of Theorem 3.4.30 (see Theorem 3.3.18), we finally get that $Q A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, v) Q'$ is of the form $\sum_{|\beta| \leq 2r} \mathscr{L}^v_{t,\beta} Z^\beta$, where $\mathscr{L}^v_{t,\beta}$ is a linear combination of operators of the type

$$Q(\lambda - \mathscr{L}^v_{3,t})^{-k'_0} R_1 (\lambda - \mathscr{L}^v_{3,t})^{-k'_1} R_2 \cdots R_{l'} (\lambda - \mathscr{L}^v_{3,t})^{-k'_{l'}} Q''' Q'', \tag{3.4.213}$$

where $\sum_j k'_j = k + l'$, $R_j \in \mathscr{F}_{t,v}$, $Q''' \in \mathcal{Q}^{2r}$ and $Q'' \in \mathcal{Q}^m$ is obtained from Q' and its commutation with Z^β. Since $k > 2(\ell + r + 1)$, we can use Proposition 3.4.30 and the arguments leading to (3.4.210) in order to split the operator in (3.4.213) into two parts:

$$\begin{aligned}
Q(\lambda - \mathscr{L}^v_{3,t})^{-k'_0} R_1 (\lambda - \mathscr{L}^v_{3,t})^{-k'_1} R_2 \cdots R_i (\lambda - \mathscr{L}^v_{3,t})^{-k''_i} \times \\
(\lambda - \mathscr{L}^v_{3,t})^{-(k'_i - k''_i)} R_{i+1} \cdots R_{l'} (\lambda - \mathscr{L}^v_{3,t})^{-k'_{l'}} Q''' Q'',
\end{aligned} \tag{3.4.214}$$

such that each part is bounded in $\| \cdot \|^{0,0}$-norm by $C(1 + |\lambda|^2)^N$. This conclude the proof of Theorem 3.4.31. □

Theorem 3.4.32. *For any $r \geq 0$ and $k > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for $\lambda \in \Gamma$, $t > 0$ and $v \geq 0$,*

$$\begin{aligned}
&\Big\| \Big(\frac{\partial^r \mathscr{L}^v_{3,t}}{\partial v^r} - \frac{\partial^r \mathscr{L}^v_{3,t}}{\partial v^r}\Big|_{v=0} \Big) s \Big\|_{-1} \leq Cv \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_1, \\
&\Big\| \Big(\frac{\partial^r}{\partial v^r} (\lambda - \mathscr{L}^v_{3,t})^{-k} - \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,r}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, 0) \Big) s \Big\|_0 \leq Cv(1+|\lambda|)^N \sum_{|\alpha| \leq 4r+3} \|Z^\alpha s\|_0.
\end{aligned} \tag{3.4.215}$$

Proof. As in the proof of Theorem 3.3.20, the first line of (3.4.215) just follows from a Taylor expansion in v of $\mathscr{L}^v_{3,t}$ and the fact that this expansion is uniform in $t > 0$. We also get an analogue of (3.3.110):

$$\Big\| ((\lambda - \mathscr{L}^v_{3,t})^{-1} - (\lambda - \mathscr{L}^0_{3,t})^{-1}) s \Big\|_0 \leq Cv(1+|\lambda|^2)^M \sum_{|\alpha| \leq 3} ||Z^\alpha s||_0. \tag{3.4.216}$$

Moreover, using Propositions 3.4.28 and 3.4.30, and (3.4.216), we have for any $m \in \mathbb{N}^*$

$$\begin{aligned}
&\Big\| ((\lambda - \mathscr{L}^v_{3,t})^{-m} - (\lambda - \mathscr{L}^0_{3,t})^{-m}) s \Big\|_0 \\
&\qquad = \Big\| \sum_{i=0}^{m-1} (\lambda - \mathscr{L}^v_{3,t})^{-i} ((\lambda - \mathscr{L}^v_{3,t})^{-1} - (\lambda - \mathscr{L}^0_{3,t})^{-1}) (\lambda - \mathscr{L}^0_{3,t})^{-(m-i-1)} s \Big\| \\
&\qquad \leq Cv(1+|\lambda|^2)^M \sum_{|\alpha| \leq 3} ||Z^\alpha s||_0.
\end{aligned} \tag{3.4.217}$$

For $(\mathbf{k}, \mathbf{r}) \in I_{k,r}$, set $a_i = (\lambda - \mathscr{L}^v_{3,t})^{-k_i}$, $b_i = \frac{\partial^{r_i} \mathscr{L}^v_{3,t}}{\partial v^{r_i}}$, $a'_i = (\lambda - \mathscr{L}^0_{3,t})^{-k_i}$ and $b'_i = \frac{\partial^{r_i} \mathscr{L}^v_{3,t}}{\partial v^{r_i}}\Big|_{v=0}$. Then

$$\begin{aligned}
A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, v) - A^{\mathbf{k}}_{\mathbf{r}}(\lambda, t, 0) &= a_0 b_1 a_1 \cdots b_j a_j - a'_0 b'_1 a'_1 \cdots b'_j a'_j \\
&= \sum_{i=1}^{j} a_0 b_1 \cdots a_{i-1} (b_i - b'_i) a'_i \cdots b'_j a'_j + \sum_{i=0}^{j} a_0 b_1 \cdots b_i (a_i - a'_i) b'_{i+1} \cdots b'_j a'_j.
\end{aligned} \tag{3.4.218}$$

Using this and (3.4.207), the first inequality of (3.4.215) and (3.4.217), we find the second inequality of (3.4.215). □

Theorem 3.4.33. *For any $\ell, \ell', r \in \mathbb{N}$ and $q > 0$, there is $C > 0$ such that for $t > 0$, $v \geq 0$ and $Z, Z' \in T_{\mathbb{R},x_0}X$ with $|Z|, |Z'| \leq q$, we have*

$$\sup_{|\alpha|, |\alpha'| \leq \ell} \Big\| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial v^r} e^{-\mathscr{L}^v_{3,t}}(Z, Z') \Big\|_{\mathscr{C}^{\ell'}(M, \mathrm{pr}^*_X \mathrm{End}(\mathbb{E}_p))} \leq C. \tag{3.4.219}$$

Proof. Using the integral representation

$$\frac{\partial^r}{\partial v^r} e^{-\mathscr{L}^v_{3,t}} = \frac{(-1)^k (k-1)!}{2i\pi} \int_\Gamma e^{-\lambda} \frac{\partial^r}{\partial v^r} (\lambda - \mathscr{L}^v_{3,t})^{-1} d\lambda, \tag{3.4.220}$$

Theorem 3.4.33 is proved from (3.4.207) and Theorem 3.4.31 exactly as Theorem 3.4.18 is proved from (3.4.136). □

For k large enough, set

$$\begin{aligned}\mathscr{B}_{r,t} &= \frac{(-1)^k(k-1)!}{2i\pi r!}\int_\Gamma e^{-\lambda}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t,0)d\lambda,\\ \mathscr{B}_{r,t,v} &= \frac{1}{r!}\frac{\partial^r}{\partial v^r}e^{-\mathscr{L}_{3,t}^v} - \mathscr{B}_{r,t}.\end{aligned} \tag{3.4.221}$$

Then $\mathscr{B}_{r,t}$ and $\mathscr{B}_{r,t,v}$ do not depend on the choice on k large. We denote by $\mathscr{B}_{r,t}(Z,Z')$ (resp. $\mathscr{B}_{r,t,v}(Z,Z')$) the smooth kernerl of $\mathscr{B}_{r,t}$ (resp. $\mathscr{B}_{r,t,v}$) with respect to $dv_{TX}(Z')$.

Theorem 3.4.34. *For $r\in\mathbb{N}$ and $q>0$, there exists $C>0$ such that for $t>0$, $v\geq 0$ and $Z,Z'\in T_{\mathbb{R},x_0}X$ with $|Z|,|Z'|\leq q$, we have*

$$\|\mathscr{B}_{r,t,v}(Z,Z')\| \leq Cv^{1/(2n_X+1)}. \tag{3.4.222}$$

Proof. The proof is the same as the proof of Theorem 3.3.21, using Theorem 3.4.32 and (3.4.220) instead of Theorem 3.3.20 and (3.3.100) respectively. ☐

Theorem 3.4.35. *For any $\ell,\ell',k\in\mathbb{N}$ and $q>0$, there is $C>0$ such that for $t>0$, $v\geq 0$ and $Z,Z'\in T_{\mathbb{R},x_0}X$ with $|Z|,|Z'|\leq q$, we have*

$$\sup_{|\alpha|,|\alpha'|\leq\ell}\left\|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial v^r}\Big(e^{-\mathscr{L}_{3,t}^v}(Z,Z') - \sum_{r=0}^k \mathscr{B}_{r,t}v^r\Big)(Z,Z')\right\|_{\mathscr{C}^{\ell'}(M,\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p))} \leq Cv^{k+1}. \tag{3.4.223}$$

Proof. By (3.4.221) and (3.4.222), we have

$$\frac{1}{r!}\frac{\partial^r}{\partial v^r}e^{-\mathscr{L}_{3,t}^v}\Big|_{v=0} = \mathscr{B}_{r,t}. \tag{3.4.224}$$

Now by Theorem 3.4.33, (3.4.221) and the Taylor expansion

$$f(v) - \sum_{r=0}^k \frac{1}{r!}\frac{\partial^r f}{\partial v^r}(0)v^r = \frac{1}{k!}\int_0^v (v-v_0)^k\frac{\partial^{k+1}f}{\partial v^{k+1}}(v_0)dv_0, \tag{3.4.225}$$

we get (3.4.223). ☐

Now, by (3.4.197) and the asymptotic expansion heat kernels (see [2] for instance), we know that $e^{-\mathscr{L}_{3,t}^v}(0,0)$ has an asymptotic expansion as $v=\sqrt{u}\to 0$ in powers of u, so we have

$$\mathscr{B}_{2r+1,t}(0,0) = 0. \tag{3.4.226}$$

Remark 3.4.36. Using ψ defined by $(\psi s)(Z) = s(-Z)$ and the asymptotic expansion of $\mathscr{L}_{3,t}^v$ as $v\to 0$, one can in fact prove directly that

$$\mathscr{B}_{r,t}(Z,Z') = (-1)^r\mathscr{B}_{r,t}(-Z,-Z'). \tag{3.4.227}$$

Theorem 3.4.35, along with (3.4.195), (3.4.197) and (3.4.226), yields to

$$\left\|u^n e^{-u\mathscr{L}_t}(0,0) - \sum_{r=0}^k \mathscr{B}_{2r,t}(0,0)u^r\right\|_{\mathscr{C}^{\ell'}(M,\mathrm{pr}_X^*\mathrm{End}(\mathbb{E}_p))} \leq Cu^{k+1}. \tag{3.4.228}$$

Thus, by the analogue of (3.3.122), we have uniformly in p

$$p^{-n_X}\psi_{1/\sqrt{p}}e^{-B^2_{p,u/p}}(x_0,x_0)=\psi_{1/\sqrt{u}}e^{-u\mathscr{L}_t}(0,0)$$
$$=\psi_{1/\sqrt{u}}\sum_{r=0}^{k}\mathscr{B}_{2r,t}(0,0)u^{r-n}+O(u^{k+1}). \tag{3.4.229}$$

In conclusion, we have proved (3.4.177) with

$$A_{p,j}=\sum_{r-\alpha=j+n}\mathscr{B}_{2r,t}(0,0)^{(2\alpha)}. \tag{3.4.230}$$

We now prove (3.4.178). To do so, we fixe $r\in\mathbb{N}$ and study the asymptotic as $t\to 0$ of $\mathscr{B}_{2r,t}(0,0)$.

We define $\underline{\mathscr{L}}^v_{3,t}$, $\underline{A}^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,v)$ and $\underline{\mathscr{B}}_{2r,t}$ to be the objects corresponding to $\mathscr{L}^v_{3,t}$, $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,v)$ and $\mathscr{B}_{2r,t}$ above when we replace $\mathscr{L}_t$ by $\underline{\mathscr{L}}_t$ in their definitions. Then all Theorems 3.4.31-3.4.35 also hold for this underlined objects. Similarly to Theorems 3.4.32 and 3.4.34, we can prove the following two results.

Theorem 3.4.37. *For any $r\geq 0$ and $k>0$, there exist $C>0$ and $N\in\mathbb{N}$ such that for $\lambda\in\Gamma$ and $t>0$,*

$$\begin{aligned}&\Big\|\Big(\frac{\partial^r\mathscr{L}^v_{3,t}}{\partial v^r}\Big|_{v=0}-\frac{\partial^r\underline{\mathscr{L}}^v_{3,t}}{\partial v^r}\Big|_{v=0}\Big)s\Big\|_{-1}\leq Ct\sum_{|\alpha|\leq r+3}\|Z^\alpha s\|_1,\\&\Big\|\Big(\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}}a^{\mathbf{k}}_{\mathbf{r}}A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,0)-a^{\mathbf{k}}_{\mathbf{r}}\underline{A}^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,0)\Big)s\Big\|_0\leq Ct(1+|\lambda|)^N\sum_{|\alpha|\leq 4r+3}\|Z^\alpha s\|_0.\end{aligned} \tag{3.4.231}$$

Theorem 3.4.38. *For $r\in\mathbb{N}$ and $q>0$, there exists $C>0$ such that for $t>0$ and $Z,Z'\in T_{\mathbb{R},x_0}X$ with $|Z|,|Z'|\leq q$, we have*

$$\|(\mathscr{B}_{r,t}-\underline{\mathscr{B}}_{r,t})(Z,Z')\|\leq Ct^{1/(2n_X+1)}. \tag{3.4.232}$$

Recall that $\mathscr{H}_{x_0}(y)$, $y\in Y_{x_0}$, is defined in (3.4.153). Once again, we define $\mathscr{H}^v_{x_0,3}(y)$, $\widetilde{A}^{\mathbf{k}}_{\mathbf{r}}(\lambda,v)(y)$ and $\widetilde{\mathscr{B}}_{2r}(y)$ to be the objects corresponding to $\mathscr{L}^v_{3,t}$, $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,v)$ and $\mathscr{B}_{2r,t}$ above when we replace $\mathscr{L}_t$ by $\mathscr{H}_{x_0}(y)$ in their definitions. Then, once again, Theorems 3.4.31-3.4.35 also hold for this objects.

By (3.4.155), we then have

$$\underline{\mathscr{L}}^v_{3,t}=P_{p,x_0}\mathscr{H}^v_{x_0,3}(\cdot)P_{p,x_0}. \tag{3.4.233}$$

As $\Delta^{T_{\mathbb{R},x_0}X}$ commutes with P_{p,x_0}, we have $(\lambda-P_{p,x_0}\Delta^{T_{\mathbb{R},x_0}X}P_{p,x_0})^{-1}=P_{p,x_0}(\lambda-\Delta^{T_{\mathbb{R},x_0}X})^{-1}P_{p,x_0}$. As a consequence, using (3.4.206) and the same reasoning as for (3.4.166) (in particular Theorem 3.4.5 and Remark 3.4.22), we find that for any $(\mathbf{k},\mathbf{r})\in I_{k,r}$, there exist $C>0$ and $K\in\mathbb{N}$ such that for the operator norm

$$\Big\|\underline{A}^{\mathbf{k}}_{\mathbf{r}}(\lambda,t,0)-P_{p,x_0}\widetilde{A}^{\mathbf{k}}_{\mathbf{r}}(\lambda,0)(\cdot)P_{p,x_0}\Big\|\leq Cp^{-1}(1+|\lambda|^2)^K. \tag{3.4.234}$$

Thus by (3.4.221),

$$\Big\|\underline{\mathscr{B}}_{2r,t}-P_{p,x_0}\widetilde{\mathscr{B}}_{2r}P_{p,x_0}\Big\|\leq Cp^{-1}. \tag{3.4.235}$$

As the proof of Theorem 3.3.21, this implies that for the operator norm,

$$\underline{\mathscr{B}}_{2r,t}(0,0)=P_{p,x_0}\widetilde{\mathscr{B}}_{2r}(0,0)P_{p,x_0}+O(p^{-1/(2n_X+1)}). \tag{3.4.236}$$

Recall that A_j is defined in (3.4.174) and (3.4.176). With the same reasoning which led to (3.4.230), we find

$$A_j = \sum_{r-\alpha=j+n} \tilde{\mathscr{B}}_{2r}(0,0)^{(2\alpha)}. \tag{3.4.237}$$

With (3.4.230), (3.4.236) and (3.4.237), we find (3.4.178) for the $\mathscr{C}^0$-norm.

Finally, using the fact that $\nabla_U^{\mathrm{pr}_M^* \mathrm{End}(\mathbb{E}_p)} \mathscr{L}_{3,t}^v$ has the same structure as $\mathscr{L}_{3,t}^v$, we can show that all the estimates in this section also hold for the derivatives of the operators involved. Thus, (3.4.178) holds for the $\mathscr{C}^\ell$-norm.

The proof of Theorem 3.4.24 is completed.

3.4.9 Proof of Theorem 3.4.26

As in (3.3.28), we can decompose

$$B_p^2 = D_p^2 + R_p, \tag{3.4.238}$$

where R_p is an operator of order 1 and of positive degree in $\Lambda^\bullet(T_\mathbb{R}^* B)$. As in (3.3.29), this implies that

$$\mathrm{Sp}(B_p^2) = \mathrm{Sp}(D_p^2). \tag{3.4.239}$$

Let

$$C_p = \frac{1}{p} B_p^2 = \frac{1}{p}(D_p^2 + R_p). \tag{3.4.240}$$

As $B_{p,u}^2 = u\psi_{1/\sqrt{u}} B_p^2 \psi_{\sqrt{u}}$, we have

$$p^{-n_Z} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} e^{-B_{p,u/p}^2} \right] = p^{-n_Z} \mathrm{Tr}_s \left[N_u \psi_{1/\sqrt{u}} e^{-uC_p} \psi_{\sqrt{u}} \right]. \tag{3.4.241}$$

By (3.4.7) and (3.4.239), there exists $\nu > 0$ such that for p large

$$\begin{aligned} &\mathrm{Sp}(D_p/\sqrt{p}) \subset]-\infty, -\sqrt{\nu}\,] \cup \{0\} \cup [\sqrt{\nu}, +\infty[, \\ &\mathrm{Sp}(C_p) \subset \{0\} \cup [\nu, +\infty[. \end{aligned} \tag{3.4.242}$$

In the sequel, we will assume that (3.4.242) holds for $p \geq 1$. Recall that δ and Δ are contours in $\mathbb{C}$ defined in Figure 3.3 in Section 3.3.6.

Set

$$\begin{aligned} \mathbb{P}_{p,u} &= \frac{1}{2i\pi} \psi_{1/\sqrt{u}} \int_\delta e^{-u\lambda} (\lambda - C_p)^{-1} d\lambda, \\ \mathbb{K}_{p,u} &= \frac{1}{2i\pi} \psi_{1/\sqrt{u}} \int_\Delta e^{-u\lambda} (\lambda - C_p)^{-1} d\lambda. \end{aligned} \tag{3.4.243}$$

Then

$$p^{-n_X} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} e^{-B_{p,u/p}^2} \right] = p^{-n_X} \mathrm{Tr}_s \left[N_u (\mathbb{P}_{p,u} + \mathbb{K}_{p,u}) \right]. \tag{3.4.244}$$

We will deal separately with the terms $\mathbb{P}_{p,u}$ and $\mathbb{K}_{p,u}$.

The term involving $\mathbb{K}_{u,p}$

For $A \in \Lambda^\bullet(T^*_\mathbb{R}B) \otimes \mathrm{End}\left(\Omega^{0,\bullet}(X, \xi \otimes F_p)\right)$, recall that $||A||_\infty$ is the norm of operator of A viewed as an endomorphism of $L^2(X, \mathbb{E}_p)$ and let

$$||A||_q = \left(\mathrm{Tr}\left[(A^*A)^{q/2}\right]\right)^{1/q}. \tag{3.4.245}$$

Remark 3.4.39. Here, we do not specify the dependance in p of the norm $||\cdot||_q$, but the reader should be aware of it.

Lemma 3.4.40. *Let $\lambda_0 \in \mathbb{R}^*_-$. Then there exists q_0 such that for $q \geq q_0$, for $U \in T_\mathbb{R}B$ and $\ell \in \mathbb{N}$, there is a $C > 0$ such that for $p \geq 1$*

$$p^{-n_X}\left\|\left(\nabla_U^{\pi^*\mathrm{End}(\mathbb{E}_p)}\right)^\ell(\lambda_0 - C_p)^{-q}\right\|_1 \leq C. \tag{3.4.246}$$

Proof. As in (3.3.199), we find using $H_p = D_p^2/p - \lambda_0$ that

$$p^{-n_X}\left\|(\lambda_0 - D_p^2/p)^{-q}\right\|_1 \leq C. \tag{3.4.247}$$

A closer look at Bismut's Lichnerowicz formula (3.2.34) and (3.2.35) enables us to sharpen (3.4.238): locally, under the trivialization on U_{x_k} (see Sections 3.3.1 and 3.4.5), we have

$$\frac{1}{p}R_p = \frac{1}{p}\mathcal{O}_{1,p} + \mathcal{O}_{0,p}, \tag{3.4.248}$$

were $\mathcal{O}_{k,p}$ is an operator of order k acting on $\mathbb{E}_p$ with bounded coefficients (with respect to the operator norm). Thus,

$$\begin{aligned} ||\mathcal{O}_{1,p}s||_{\boldsymbol{H}^k(p)} &\leq C||s||_{\boldsymbol{H}^{k+1}(p)}, \\ ||\mathcal{O}_{0,p}s||_{\boldsymbol{H}^k(p)} &\leq C||s||_{\boldsymbol{H}^k(p)}. \end{aligned} \tag{3.4.249}$$

On the other hand, locally and in the spirit of (3.3.8) and (3.4.89),

$$D_p = D^X + pM_p \tag{3.4.250}$$

where M_p is a uniformly bounded operator of order 0 acting on $\mathbb{E}_p$. As D^X is elliptic, this yields to

$$||s||_{\boldsymbol{H}^1(p)} \leq C\left(||D_ps||_{L^2} + p||s||_{L^2}\right). \tag{3.4.251}$$

Consequently, if s is an eigenfunction of $D_p/\sqrt{p}$ for the eigenvalue μ, (3.4.248) and (3.4.249) imply

$$\begin{aligned} \frac{1}{p}||R_ps||_{L^2} &\leq \frac{1}{p}||s||_{\boldsymbol{H}^1(p)} + ||s||_{L^2} \\ &\leq C\frac{1}{p}||D_ps||_{L^2} + C'||s||_{L^2} \\ &\leq C\Big(1 + \frac{|\mu|}{\sqrt{p}}\Big)||s||_{L^2} \leq C(1 + |\mu|)||s||_{L^2}. \end{aligned} \tag{3.4.252}$$

Hence,

$$\frac{1}{p}\left\|R_p(\lambda_0 - D_p^2/p)^{-1}\right\|_\infty \leq C \sup_{\mu \in [\sqrt{\nu}, +\infty[} \frac{1+\mu}{|\lambda_0 - \mu^2|} \leq C'. \tag{3.4.253}$$

As in (3.3.30), we have

$$(\lambda_0 - C_p)^{-1} = (\lambda_0 - D_p^2/p)^{-1} + (\lambda_0 - D_p^2/p)^{-1}(R_p/p)(\lambda_0 - D_p^2/p)^{-1} + \cdots , \quad (3.4.254)$$

with only finitely many terms (as R_p is sum of element of positive degree in $\Lambda^\bullet(T^*_{\mathbb{R}}B)$). With this expansion, (3.3.191) (3.4.247) and (3.4.253) we get Lemma 3.4.40 for $\ell = 0$ as we did in Lemma 3.3.29.

For $\ell \geq 1$, the reasoning in the same as in Lemma 3.3.29. □

Proposition 3.4.41. *For any $\ell \in \mathbb{N}$, there exist $a, C > 0$ such that for $p \geq 1$ and $u \geq 1$,*

$$p^{-n_X} \big| \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] \big|_{\mathscr{C}^\ell(B)} \leq C e^{-au}. \quad (3.4.255)$$

Proof. Proposition 3.4.41 follows from Lemma 3.4.40 exactly as Proposition 3.3.30 follows from Lemma 3.3.29. □

The term involving $\mathbb{P}_{p,u}$

Proposition 3.4.42. *For any $\ell \in \mathbb{N}$, there is a $C > 0$ such that for any $p \geq 1$ and $u \geq 1$,*

$$p^{-n_Z} \big| \operatorname{Tr}_s [N_u \mathbb{P}_{p,u}] \big|_{\mathscr{C}^\ell(B)} \leq \frac{C}{\sqrt{u}}. \quad (3.4.256)$$

Proof. The proof is exactly the same as the proof of Proposition 3.3.31, the only changes is that to prove the analogue of (3.3.242), we use (3.4.252) instead of its analogue (3.3.203), and that we substitute the estimate in (3.3.243) by

$$p^{-n_Z} \dim \ker(D_p^2) = p^{-n_Z} \dim H^0(X, \xi \otimes F_p) = p^{-n_Z} \dim H^0(Z, \pi_1^*\xi \otimes \eta \otimes L^p) \leq C. \quad (3.4.257)$$

□

With (3.4.244) and Propositions 3.4.41 and 3.4.42, we have proved Theorem 3.4.26.

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