Variance of the Volume of Random Real Algebraic Submanifolds II

THOMAS LETENDRE & MARTIN PUCHOL

ABSTRACT. Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals, and let M be its real locus. We study the vanishing locus Z_{s_d} in M of a random real holomorphic section s_d of $\mathcal{E} \otimes \mathcal{L}^d$, where $\mathcal{L} \to \mathcal{X}$ is an ample line bundle and $\mathcal{E} \to \mathcal{X}$ is a rank r Hermitian bundle, $r \in \{1, \dots, n\}$. We establish the asymptotic of the variance of the linear statistics associated with Z_{s_d}, as d goes to infinity. This asymptotic is of order $d^{r-n/2}$. As a special case, we get the asymptotic variance of the volume of Z_{s_d}.

The present paper extends the results of [22], by the first-named author, in essentially two ways. First, our main theorem covers the case of maximal codimension ($r = n$), which was left out in [22]. Second, we show that the leading constant in our asymptotic is positive. This last result is proved by studying the Wiener-Itô expansion of the linear statistics associated with the common zero set in $\mathbb{R}\mathbb{P}^n$ of r independent Kostlan-Shub-Smale polynomials.

CONTENTS

1. INTRODUCTION

In recent years, the study of random submanifolds has been a very active research area [9, 14, 15, 25, 26, 29]. There exist several models of random submanifolds, built on the following principle. Given M a dimension n ambient manifold and $r \in \{1, \dots, n\}$, we consider the common zero set of r independent random functions on M. Under some technical assumption, this zero set is almost surely a codimension r smooth submanifold.

In this paper, we are interested in a model of random real algebraic submanifolds in a projective manifold. It was introduced in this generality by Gayet and Welschinger in [13] and studied in [14, 15, 21, 22], among others. This model is the real counterpart of the random complex algebraic submanifolds considered by Bleher, Shiffman and Zelditch [6, 30, 31].

Framework. Let us describe more precisely our framework. More details are given in Section 2, below. Let $\mathcal{X}$ be a smooth complex projective manifold of dimension $n \geqslant 1$. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$, and let $\mathcal{E}$ be a rank $r \in \{1, \dots, n\}$ holomorphic vector bundle over $\mathcal{X}$. We assume that $\mathcal{X}$, $\mathcal{E}$, and $\mathcal{L}$ are endowed with compatible real structures and that the real locus of $\mathcal{X}$ is not empty. We denote by M this real locus which is a smooth closed (i.e., compact without boundary) manifold of real dimension n.

Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on $\mathcal{E}$ and $\mathcal{L}$, respectively, which are compatible with the real structures. We assume $h_{\mathcal{L}}$ has positive curvature ω, so that ω is a Kähler form on $\mathcal{X}$. This ω induces a Riemannian metric g on $\mathcal{X}$, and hence on M. Let us denote by $|\mathrm{d}V_M|$ the Riemannian volume measure on M induced by g.

For any $d \in \mathbb{N}$, the measure $|\mathrm{d}V_M|$ and the metrics $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ induce a Euclidean inner product on the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ (see equation (2.2)). Given $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ the real zero set of s. For d large enough, for almost every s with respect to the Lebesgue measure, Z_s is a codimension r smooth closed submanifold of M, possibly empty. We denote by $|\mathrm{d}V_s|$ the Riemannian volume measure on Z_s induced by g. In the following, we consider $|\mathrm{d}V_s|$ as a Radon measure on M, which is a continuous linear form on $(C^0(M), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the sup norm.

Remark 1.1. If $n = r$ then Z_s is a finite subset of M for almost every s. In this case, $|\mathrm{d}V_s|$ is the sum of the unit Dirac masses on the points of Z_s.

Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, $|\mathrm{d}V_{s_d}|$ is a random positive Radon measure on M. We set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$ in order to simplify notation. We are interested in the asymptotic distribution of the linear statistics $\langle |\mathrm{d}V_d|, \phi\rangle = \int_{Z_d} \phi \, |\mathrm{d}V_d|$, where $\phi : M \to \mathbb{R}$ is a continuous test-function. In particular, $\langle |\mathrm{d}V_d|, \mathbf{1}\rangle$ is the volume of Z_d (its cardinal if $n = r$), where $\mathbf{1}$ is the unit constant function on M.

As usual, we denote by $\mathbb{E}[X]$ the mathematical expectation of the random vector X. The asymptotic expectation of $\langle |\mathrm{d}V_d|, \phi\rangle$ was computed in Section 5.3 of [21].

Theorem 1.2 ([21]). *Let $\mathcal{X}$ be a complex projective manifold of positive dimension n defined over the reals; we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ Hermitian vector bundle, and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, the following holds as $d \to +\infty$:*

$$\forall\, \phi \in C^0(M), \quad \mathbb{E}[\langle |\mathrm{d}V_d|, \phi\rangle] = d^{r/2}\left(\int_M \phi |\mathrm{d}V_M|\right)\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)} + \|\phi\|_\infty O(d^{r/2-1}).$$

Moreover, the error term $O(d^{r/2-1})$ does not depend on ϕ.

The asymptotic variance of $\langle |\mathrm{d}V_d|, \phi\rangle$, as d goes to infinity, was proved to be a $O(d^{r-n/2})$ when the codimension of Z_d is $r < n$ (see [22, Theorem 1.6]). Our first main theorem (Theorem 1.6 below) extends this result to the maximal codimension case.

Statement of the main results. Before we state our main result, let us introduce some more notation. We denote the covariance of the real random variables X and Y by $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$. Let $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$ denote the variance of X. Finally, we call *variance* of $|\mathrm{d}V_d|$, and we denote by $\mathrm{Var}(|\mathrm{d}V_d|)$ the symmetric bilinear form on $C^0(M)$ defined by

$$\forall\, \phi_1, \phi_2 \in C^0(M), \quad \mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = \mathrm{Cov}(\langle |\mathrm{d}V_d|, \phi_1\rangle, \langle |\mathrm{d}V_d|, \phi_2\rangle).$$

Definition 1.3. Given $\phi \in C^0(M)$, we denote by ϖ_ϕ its *continuity modulus*, which is the function from $(0, +\infty)$ to $[0, +\infty)$ defined by

$$\varpi_\phi : \varepsilon \longmapsto \sup\{|\phi(x) - \phi(y)| \,\big|\, (x, y) \in M^2,\ \rho_g(x, y) \leqslant \varepsilon\},$$

where $\rho_g(\cdot, \cdot)$ stands for the geodesic distance on (M, g).

We denote by $\mathcal{M}_{rn}(\mathbb{R})$ the space of matrices of size $r \times n$ with real coefficients.

Definition 1.4. Let $L : V \to V'$ be a linear map between two Euclidean spaces. We denote the *Jacobian* of L by $|\det{}^\perp(L)| = \sqrt{\det(LL^*)}$, where $L^* : V' \to V$ is the adjoint operator of L. Similarly, let $A \in \mathcal{M}_{rn}(\mathbb{R})$; we define its *Jacobian* to be $|\det{}^\perp(A)| = \sqrt{\det(AA^t)}$.

Definition 1.5. For every $t > 0$, we define $(X(t), Y(t))$ to be a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ such that the following hold:

- The couples $(X_{ij}(t), Y_{ij}(t))$ with $i, j \in \{1, \dots, n\}$ are independent from one another.

- The variance matrix of $(X_{ij}(t), Y_{ij}(t))$ is

$$\begin{pmatrix} 1 - \dfrac{te^{-t}}{1-e^{-t}} & e^{-t/2}\left(1 - \dfrac{t}{1-e^{-t}}\right) \\ e^{-t/2}\left(1 - \dfrac{t}{1-e^{-t}}\right) & 1 - \dfrac{te^{-t}}{1-e^{-t}} \end{pmatrix} \quad \text{if } j = 1,$$

and

$$\begin{pmatrix} 1 & e^{-t/2} \\ e^{-t/2} & 1 \end{pmatrix} \quad \text{otherwise.}$$

We can now state our main result.

Theorem 1.6. *Let $\mathcal{X}$ be a complex projective manifold of dimension $n \geqslant 1$ defined over the reals. We assume its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ Hermitian vector bundle, and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.*

Let $\beta \in (0, \frac{1}{2})$; then, there exists $C_\beta > 0$ such that, for all $\alpha \in (0,1)$, for all ϕ_1 and $\phi_2 \in C^0(M)$, the following holds as $d \to +\infty$:

$$\begin{aligned} \text{(1.1)} \quad & \operatorname{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) \\ &= d^{r-n/2}\left(\int_M \phi_1\phi_2 |\mathrm{d}V_M|\right)\left(\frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r}\mathcal{I}_{n,r} + \delta_{rn}\frac{2}{\operatorname{Vol}(\mathbb{S}^n)}\right) \\ &\quad + \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{r-n/2-\alpha}) + \|\phi_1\|_\infty \varpi_{\phi_2}(C_\beta d^{-\beta}) O(d^{r-n/2}), \end{aligned}$$

where δ_{rn} is the Kronecker symbol, equal to 1 if $r = n$ and 0 otherwise, and

$$\begin{aligned} \text{(1.2)} \quad \mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} \Bigg(& \frac{\mathbb{E}[|\det{}^\perp(X(t))|\,|\det{}^\perp(Y(t))|]}{(1-e^{-t})^{r/2}} \\ & - (2\pi)^r\left(\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\right)^2\Bigg) t^{(n-2)/2}\,\mathrm{d}t < +\infty. \end{aligned}$$

Moreover, the error terms $O(d^{r-n/2-\alpha})$ and $O(d^{r-n/2})$ in (1.1) do not depend on (ϕ_1, ϕ_2).

Remark 1.7. Applying Theorem 1.6 with $\phi_1 = \mathbf{1} = \phi_2$ gives the asymptotic variance of the Riemannian volume of Z_d.

Theorem 1.8. *For any $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, the universal constant*

$$\frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r}\mathcal{I}_{n,r} + \delta_{rn}\frac{2}{\operatorname{Vol}(\mathbb{S}^n)}$$

appearing in Theorem 1.6 is positive.

Remark 1.9. Theorem 1.8 was proved for $r = n = 1$ in [11], and for $r = n \geqslant 2$ in [2]. Note that Theorem 1.8 states that $\mathcal{I}_{n,r} > 0$ if $r < n$, but this is not necessarily the case when $r = n$. Indeed, $\mathcal{I}_{1,1} < 0$ by Proposition 3.1 and Remark 1 in [11].

Let us state some corollaries of Theorem 1.6. Corollary 1.10, 1.11, and 1.12 below are extensions to the case $r \leqslant n$ of Corollary 1.9, 1.10, and 1.11 of [22], respectively. The proofs that Theorem 1.6 implies (i.e., Corollaries 1.10, 1.11, and 1.12) were given in [22, Section 5] in the case $r < n$; they are still valid for $r \leqslant n$. We do not reproduce these proofs in the present paper.

Corollary 1.10 (Concentration in probability). *In the same setting as Theorem 1.6, let $\alpha > -n/4$ and let $\phi \in C^0(M)$. Then, for every $\varepsilon > 0$, we have*

$$\mathbb{P}\left(d^{-r/2}\left|\langle |\mathrm{d}V_d|, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d|, \phi\rangle]\right| > d^{\alpha}\varepsilon\right) = \frac{1}{\varepsilon^2} O(d^{-(n/2+2\alpha)}),$$

where the error term is independent of ε, but depends on ϕ.

Corollary 1.11. *In the same setting as Theorem 1.6, let $U \subset M$ be an open subset; then, as $d \to +\infty$ we have $\mathbb{P}(Z_d \cap U = \emptyset) = O(d^{-n/2})$.*

Let us denote the standard Gaussian measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ by $\mathrm{d}\nu_d$ (see (2.1)). Let $\mathrm{d}\nu$ denote the product measure $\bigotimes_{d \in \mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d \in \mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, we have the following result.

Corollary 1.12 (Strong law of large numbers). *In the setting of Theorem 1.6, let us assume $n \geqslant 3$. Let $(s_d)_{d \in \mathbb{N}} \in \prod_{d \in \mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a random sequence of sections. Then, $\mathrm{d}\nu$-almost surely,*

$$d^{-r/2}|\mathrm{d}V_{s_d}| \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)} |\mathrm{d}V_M|,$$

in the sense of the weak convergence of measures. That is, $\mathrm{d}\nu$-almost surely,

$$\forall \phi \in C^0(M), \quad d^{-r/2}\langle |\mathrm{d}V_{s_d}|, \phi\rangle \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)} \left(\int_M \phi |\mathrm{d}V_M|\right).$$

Related works and novelty of the main results. This paper extends the results of [22] by the first-named author. In [22, Theorem 1.6], our main result (Theorem 1.6 above) was proved for $r < n$ and $\alpha \in (0, \alpha_0)$, where $\alpha_0 \in (0, 1)$ is some explicit constant depending on n and r. The main novelty in Theorem 1.6 is that it covers the case of maximal codimension ($r = n$), that is, the case where Z_d is almost surely a finite subset of M. This case was not considered in [22] because of additional singularities arising in the course of the proof, which caused it to fail when $r = n$.

An important contribution of the present paper is that we prove new estimates (see Lemmas 5.26, 5.28, and 5.29) for operators related to the Bergman kernel of

$\mathcal{E} \otimes \mathcal{L}^d$, which is the correlation kernel of the random field $(s_d(x))_{x \in M}$. These estimates are one of the key improvements that allow us to prove Theorem 1.6 in the case $r = n$. They also allow us to consider $\alpha \in (0, 1)$ instead of $\alpha \in (0, \alpha_0)$. Finally, the use of these estimates greatly clarifies the proof of Theorem 1.6 in the case $r < n$, compared to the proof given in [22]. For this reason, we give the proof of Theorem 1.6 in the general case $r \in \{1, \dots, n\}$ and not only for $r = n$. This does not lengthen the proof.

The second main contribution of this article is the proof of the positivity of the leading constant in Theorem 1.6 (cf. Theorem 1.8). This result did not appear in [22]. Since the leading constant in Theorem 1.6 is universal, when $r = n$ one can deduce Theorem 1.8 from results of Dalmao [11] (if $r = n = 1$) and Armentano-Azas-Dalmao-León [2] (if $r = n \geqslant 2$). In [2, 11], the authors proved Theorem 1.6 in the special case where Z_d is the zero set in $\mathbb{RP}^n$ of n independent Kostlan-Shub-Smale polynomials (see Subsection 6.1 below). Their results include the positivity of the leading constant, and hence imply Theorem1.8 in this case. To the best of our knowledge, Theorem 1.8 is completely new for $r < n$.

Note that when $n = r = 1$, our setting covers the case of the binomial polynomials on $\mathbb{C}$ with standard Gaussian coefficients. Much more is known in this case, including variance estimates for the number of real zeros of non-Gaussian ensembles of real polynomials (see [34]).

Our proof of Theorem 1.8 uses the Wiener-Itô expansion of the linear statistics associated with the field $(s_d(x))_{x \in M}$. This kind of expansion has been studied by Slud [33] and Kratz-León [18, 19]. It was used in a random geometry context in [2, 11, 12, 25]. In [12, 25], the authors used these Wiener chaos techniques to prove Central Limit Theorems for the volume of the zero set of Arithmetic Random Waves on the two-dimensional flat torus (see also [11] in an algebraic setting). In [2, 11], these methods where used to prove Theorem 1.8 when $r = n$.

In the related setting of Riemannian Random Waves, Canzani and Hanin [8] obtained recently an asymptotic upper bound for the variance of the linear statistics. To the best of our knowledge, in this Riemannian setting, the precise asymptotic of the variance of the volume of random submanifolds is known only when the ambient manifold is $\mathbb{S}^2$ (cf. [35]) or $\mathbb{T}^2$ (cf. [12, 20]). We refer to the introduction of [22] for more details about related works.

About the proofs. The proof of Theorem 1.6 broadly follows the lines of the proof of [22, Theorem 1.6]. The random section s_d defines a centered Gaussian field $(s_d(x))_{x \in M}$ whose correlation kernel is E_d, the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$ (see Subsection 2.4). Thanks to results of Dai-Liu-Ma [10] and Ma-Marinescu [24], we know that this kernel decreases exponentially fast outside of the diagonal $\Delta = \{(x, y) \in M^2 \mid x = y\}$, and that it admits a universal local scaling limit close to Δ (see Section 3 for details).

By an application of Kac-Rice formulas (cf. Theorems 5.1, 5.5), we can express the covariance of $\langle |\mathrm{d}V_d|, \phi_1 \rangle$ and $\langle |\mathrm{d}V_d|, \phi_2 \rangle$ as a double integral over $M \times M$ of $\phi_1(x)\phi_2(y)$ times a density function $\mathcal{D}_d(x, y)$ that depends only on E_d. Our main concern is to understand the asymptotic of the integral of $\mathcal{D}_d(x,y)$, as $d \to +\infty$.

Thanks to the exponential decay of the Bergman kernel, we can show that the leading term in our asymptotic is given by the integral of $\mathcal{D}_d$ over a neighborhood Δ_d of Δ, of typical size $1/\sqrt{d}$ (see Proposition 5.22). Changing variables so that we integrate on a domain of typical size independent of d leads to the apparition of a factor $d^{-n/2}$. Besides, $\mathcal{D}_d$ takes values of order d^r on Δ_d (see Proposition 5.25). This explains why the asymptotic variance is of order $d^{r-n/2}$ in Theorem 1.6.

The behavior of E_d allows us to prove that $\mathcal{D}_d$ admits a universal local scaling limit on Δ_d. The main difficulty in our proof is to show that the convergence to this scaling limit is uniform on Δ_d (see Proposition 5.25 for a precise statement). This difficulty comes from the fact that $\mathcal{D}_d$ is singular along Δ, just like almost everything in this problem. This is where our proof differs from [22]. In [22], the uniform convergence of $\mathcal{D}_d$ to its scaling limit on Δ_d is not established, and one has to work around this lack of uniformity. This yields a complicated proof that fails when $r = n$. Here, we manage to prove this uniform convergence, thanks to some new key estimates (see Lemmas 5.26, 5.28, and 5.29) that form the technical core of the paper. This allows us to both improve on the results of [22] and simplify their proof.

As we explained, our proof relies on two properties of the Bergman kernel E_d: namely, the existence of a scaling limit around any point at scale $1/\sqrt{d}$, and its exponential decay outside of the diagonal. These features are also exhibited by Bergman kernels in other settings such as those of [4] or [5], so one might hope to generalize our results to these settings, at least in the bulk. Unfortunately, we also need a precise understanding of the scaling limit of E_d, which is possible in our framework because it is universal (it only depends on n) and invariant under isometries (see Section 4 for more details). As far as we know, it is much more complicated to study this scaling limit in other settings (such as those of [4] and [5]), so we do not pursue this line of inquiry in the present paper, and leave it for future research.

Let us now discuss the proof of Theorem 1.8. One would expect to prove this by computing a good lower bound for $\mathcal{I}_{n,r}$, directly from its expression (see equation (1.2)). To the best of our knowledge this approach fails, and we have to use subtler techniques.

Since the leading constant in (1.1) only depends on n and r, we can focus on the case of the volume of Z_d (where $\phi_1 = \mathbf{1} = \phi_2$) in a particular geometric setting. We consider the common real zero set of r independent Kostlan-Shub-Smale polynomials in $\mathbb{RP}^n$ (see Subsection 6.1 for details). This allows for explicit computations since the Bergman kernel is explicitly known in this setting. Moreover, the distribution of these polynomials is invariant under the action of $O_{n+1}(\mathbb{R})$ on $\mathbb{RP}^n$, which leads to useful independence proprieties that are not satisfied in general.

In this framework, we adapt the strategy of [2, 11] to the case $r < n$. First, we compute the Wiener-Itô expansion of the volume of Z_d. That is, we expand $\mathrm{Vol}(Z_d)$ as $\sum_{q\in\mathbb{N}} \mathrm{Vol}(Z_d)[q]$, where the convergence is in the space of L^2 random variables on our probability space, and $\mathrm{Vol}(Z_d)[q]$ denotes the q-th chaotic component of $\mathrm{Vol}(Z_d)$. In particular, $\mathrm{Vol}(Z_d)[0]$ is the expectation of $\mathrm{Vol}(Z_d)$, and the $(\mathrm{Vol}(Z_d)[q])_{q\in\mathbb{N}}$ are pairwise orthogonal L^2 random variables. Hence,

$$\mathrm{Var}(\mathrm{Vol}(Z_d)) = \sum_{q\geqslant 1} \mathrm{Var}(\mathrm{Vol}(Z_d)[q]).$$

The chaotic components of odd order of $\mathrm{Vol}(Z_d)$ are zero, but we prove that $\mathrm{Var}(\mathrm{Vol}(Z_d)[2])$ is equivalent to $d^{r-n/2}C$ as $d \to +\infty$, where $C > 0$ (see Lemma 6.17). This implies Theorem 1.8.

Outline of the paper. In Section 2, we describe precisely our framework and the construction of the random measures $|\mathrm{d}V_d|$. We also introduce the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$ and prove that it is the correlation kernel of $(s_d(x))_{x\in M}$. In Section 3, we recall estimates for the Bergman kernel, and its scaling limit. Section 4 is dedicated to the study of the Bargmann-Fock process, that is, the Gaussian centered random process on $\mathbb{R}^n$ whose correlation function is

$$(w, z) \longmapsto \exp(-\frac{1}{2}\|w - z\|^2).$$

This field is the local scaling limit of the random field $(s_d(x))_{x\in M}$, in a sense to be made precise below. Section 5 and 6 are concerned with the proofs of Theorem 1.6 and Theorem 1.8, respectively. Note that in Section 6 we have to study in detail the model of Kostlan-Shub-Smale polynomials, which is the simplest example of our general real algebraic setting. We conclude this paper by two appendices, Appendix A and Appendix B, in which we have gathered the proofs of the technical lemmas of Section 4 and Section 5, respectively.

2. RANDOM REAL ALGEBRAIC SUBMANIFOLDS

In this section, we introduce the main objects we will be studying throughout this paper. We first recall some basic definitions in Subsection 2.1. In Subsection 2.2, we introduce our geometric framework. In Subsection 2.3, we describe our model of random real algebraic submanifolds. Finally, we relate these random submanifolds to Bergman kernels in Subsection 2.4.

2.1. Random vectors. Let us recall some facts about random vectors (see, e.g., [21, Appendix A]). In this paper, we only consider centered random vectors, so we give the following definitions in this setting.

Let X_1 and X_2 be centered random vectors taking values in Euclidean (or Hermitian) vector spaces V_1 and V_2, respectively; then, we define their *covariance operator* as

$$\mathrm{Cov}(X_1, X_2) : v \longmapsto \mathbb{E}[X_1 \langle v, X_2\rangle]$$

from V_2 to V_1. For all $v \in V_2$, we set $v^* = \langle \cdot, v \rangle \in V_2^*$. Then, $\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 \otimes X_2^*]$ is an element of $V_1 \otimes V_2^*$. Let X be a centered random vector in a Euclidean space V. The *variance operator* of X is defined as $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathbb{E}[X \otimes X^*] \in V \otimes V^*$. Let Λ be a non-negative self-adjoint operator on $(V, \langle \cdot, \cdot \rangle)$. We denote by $X \sim \mathcal{N}(\Lambda)$ the fact that X is a centered Gaussian vector with variance operator Λ. This means that the characteristic function of X is $\xi \mapsto \exp(-\frac{1}{2}\langle \Lambda\xi, \xi\rangle)$. Finally, we say that $X \in V$ is a *standard* Gaussian vector if $X \sim \mathcal{N}(\mathrm{Id})$, where Id is the identity operator on V.

If Λ is positive, the distribution of $X \sim \mathcal{N}(\Lambda)$ admits the density

$$x \mapsto \frac{1}{\sqrt{2\pi}^N \sqrt{\det(\Lambda)}} \exp\left(-\frac{1}{2}\langle \Lambda^{-1}x, x\rangle\right) \tag{2.1}$$

with respect to the normalized Lebesgue measure of V, where $N = \dim(V)$. Otherwise, X takes values in $\ker(\Lambda)^\perp$ almost surely, and it admits a similar density as a variable in $\ker(\Lambda)^\perp$.

2.2. General setting. Let us introduce more precisely our geometric framework. Let $\mathcal{X}$ be a smooth complex projective manifold of positive complex dimension n. Let $c_{\mathcal{X}}$ be a real structure on $\mathcal{X}$, that is, an anti-holomorphic involution. The real locus of $(\mathcal{X}, c_{\mathcal{X}})$ is the set M of fixed points of $c_{\mathcal{X}}$. In the following, we assume M is non-empty. It is known that M is a smooth closed submanifold of $\mathcal{X}$ of real dimension n (see [32, Chapter 1]).

Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle of rank $r \in \{1, \dots, n\}$. We denote by $\pi_{\mathcal{E}}$ its bundle projection. Let $c_{\mathcal{E}}$ be a real structure on $\mathcal{E}$, compatible with $c_{\mathcal{X}}$ in the sense that $c_{\mathcal{X}} \circ \pi_{\mathcal{E}} = \pi_{\mathcal{E}} \circ c_{\mathcal{E}}$ and $c_{\mathcal{E}}$ is fiberwise $\mathbb{C}$-anti-linear. Let $h_{\mathcal{E}}$ be a Hermitian metric on $\mathcal{E}$ such that $c_{\mathcal{E}}^\star(h_{\mathcal{E}}) = \overline{h_{\mathcal{E}}}$. A Hermitian metric satisfying this condition is said to be *real*. Similarly, let $\mathcal{L} \to \mathcal{X}$ be an ample holomorphic line bundle equipped with a compatible real structure $c_{\mathcal{L}}$ and a real Hermitian metric $h_{\mathcal{L}}$.

We assume that $(\mathcal{L}, h_{\mathcal{L}})$ has positive curvature, that is, its curvature form ω is Kähler. Recall that, if ζ_0 is a local non-vanishing holomorphic section of $\mathcal{L}$, then $\omega = (1/(2i))\partial\bar{\partial}\ln(h_{\mathcal{L}}(\zeta_0, \zeta_0))$ locally. This Kähler form defines a Riemannian metric g on $\mathcal{X}$ (see, e.g., [16, Section 0.2]). In turn, g induces a Riemannian volume measure on $\mathcal{X}$ and on any smooth submanifold of $\mathcal{X}$. We denote by $\mathrm{d}V_{\mathcal{X}} = \omega^n/n!$ the Riemannian volume form on $(\mathcal{X}, g)$. Similarly, let $|\mathrm{d}V_M|$ denote the Riemannian measure on (M, g).

Let $d \in \mathbb{N}$; then, we endow $\mathcal{E} \otimes \mathcal{L}^d$ with the real structure $c_d = c_{\mathcal{E}} \otimes (c_{\mathcal{L}})^d$, which is compatible with $c_{\mathcal{X}}$, and the real Hermitian metric $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$. Let $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$; we then define a Hermitian inner product on $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ by

$$\forall\, s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \quad \langle s_1, s_2 \rangle = \int_{\mathcal{X}} h_d(s_1(x), s_2(x))\mathrm{d}V_{\mathcal{X}}. \tag{2.2}$$

Remark 2.1. In this paper, $\langle \cdot, \cdot \rangle$ will either denote the inner product of a Euclidean (or Hermitian) space or the duality pairing between a Banach space and its topological dual. Which one should be clear from the context.

We say that a section $s \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is *real* if it is equivariant for the real structures, that is, $c_d \circ s = s \circ c_{\mathcal{X}}$. We denote by $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ the real vector space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product. Note that, despite their name, real sections are defined on the whole complex locus $\mathcal{X}$ and that the Euclidean inner product is defined by integrating over $\mathcal{X}$, not just M.

Let $x \in M$; then, the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$ is a dimension r complex vector space, and the restriction of c_d to this space is a $\mathbb{C}$-anti-linear involution. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the set of fixed points of this involution, which is a real r-dimensional vector space. Then, $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$ is a rank r real vector bundle, and, for any $s \in \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, the restriction of s to M is a smooth section of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$.

Let $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This space is known to be finite dimensional (compare [23, Theorem 1.4.1]). Let N_d denote the complex dimension of $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. We denote by

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \{s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \mid c_d \circ s = s \circ c_{\mathcal{X}}\}$$

the space of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. The restriction of the inner product (2.2) to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (respectively, $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$) makes it into a Euclidean (respectively, Hermitian) space of real (respectively, complex) dimension N_d.

2.3. Random real algebraic submanifolds. This section is concerned with the definition of the random submanifolds we consider and some related random variables.

Let $d \in \mathbb{N}$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, we denote by Z_s the real zero set $s^{-1}(0) \cap M$ of s. If the restriction of s to M vanishes transversally, then Z_s is a smooth closed submanifold of codimension r of M (note that this includes the case where Z_s is empty). In this case, we denote by $|\mathrm{d}V_s|$ the Riemannian volume measure on Z_s induced by g. In the following, we consider $|\mathrm{d}V_s|$ as the continuous linear form on $(C^0(M), \|\cdot\|_\infty)$ defined by

$$\forall\, \phi \in C^0(M), \qquad \langle |\mathrm{d}V_s|, \phi \rangle = \int_{x \in Z_s} \phi(x) |\mathrm{d}V_s|.$$

Definition 2.2 (***compare*** [27]). We say that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-*ample* if, for any $x \in M$, the evaluation map $\mathrm{ev}_x^d : s \mapsto s(x)$ from $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ to $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ is surjective.

Lemma 2.3. *There exists $d_1 \in \mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$, and $\mathcal{L}$, such that for all $d \geqslant d_1$, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample.*

Proof. This can be deduced from the Riemann-Roch formula, for example. It is also a byproduct of the computations of the present paper and will be proved later on (see Corollary 5.11 below). ◻

Let us now consider a random section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Recall that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, endowed with the inner product (2.2), is a Euclidean inner product of dimension N_d.

Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

Lemma 2.4. *For every $d \geqslant d_1$, Z_{s_d} is almost surely a smooth closed codimension r submanifold of M.*

Proof. Since $d \geqslant d_1$, we have that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample. By a transversality argument (see [21, Section 2.6] for details), this implies that the restriction of s to M vanishes transversally for almost every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (with respect to the Lebesgue measure). Thus, almost surely, s_d restricted to M vanishes transversally, and its zero set is a smooth closed submanifold of codimension r. ◻

From now on, we only consider the case $d \geqslant d_1$, so that Z_{s_d} is almost surely a random smooth closed submanifold of M of codimension r. For simplicity, we denote $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$. Let $\phi \in C^0(M)$ and $d \geqslant d_1$; then, the real random variable $\langle |\mathrm{d}V_d|, \phi\rangle$ is called the *linear statistic* of degree d associated with ϕ. For example, $\langle |\mathrm{d}V_d|, \mathbf{1}\rangle$ is the volume of Z_d.

2.4. The correlation kernel. The random section $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ defines a centered Gaussian process $(s_d(x))_{x\in\mathcal{X}}$, for any $d \in \mathbb{N}$. In this section, we recall the relation between the distribution of this process and the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$.

Recall that $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ stands for the bundle

$$P_1^\star(\mathcal{E} \otimes \mathcal{L}^d) \otimes P_2^\star((\mathcal{E} \otimes \mathcal{L}^d)^*)$$

over $\mathcal{X} \times \mathcal{X}$, where P_1 (respectively, P_2) denotes the projection from $\mathcal{X} \times \mathcal{X}$ onto the first (respectively, second) $\mathcal{X}$ factor. The distribution of $(s_d(x))_{x\in\mathcal{X}}$ is characterized by its covariance kernel, that is, the section of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ defined by: $(x, y) \mapsto \mathrm{Cov}(s_d(x), s_d(y)) = \mathbb{E}[s_d(x) \otimes s_d(y)^*]$.

Definition 2.5. Let E_d denote the *Bergman kernel* of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$, that is, the Schwartz kernel of the orthogonal projection from $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

Let $(s_{1,d}, \dots, s_{N_d,d})$ be an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, it is also an orthonormal basis of $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Recall that E_d is given by

$$E_d : (x, y) \longmapsto \sum_{i=1}^{N_d} s_{i,d}(x) \otimes s_{i,d}(y)^*.$$

This shows that E_d is a real global holomorphic section of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$. The following proves that the distribution of $(s_d(x))_{x\in\mathcal{X}}$ is totally described by E_d.

Proposition 2.6 (***compare*** [22, Proposition 2.6]). *Let $d \in \mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, for all x and $y \in \mathcal{X}$, we have* $\mathrm{Cov}(s_d(x), s_d(y)) = E_d(x, y)$.

Thus, the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$ gives the correlations between the values of the random section s_d. By taking partial derivatives of this relation, we obtain the correlations between the values of s_d and its derivatives. More details about what follows can be found in [22, Section 2.3].

Let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$. This induces a dual connection $(\nabla^d)^*$ on $(\mathcal{E} \otimes \mathcal{L}^d)^*$, which is compatible with the metric (cf. [16, Section 0.5]). We can then define a natural metric connection ∇_1^d on $P_1^\star(\mathcal{E} \otimes \mathcal{L}^d) \to \mathcal{X} \times \mathcal{X}$ whose partial derivatives are ∇^d with respect to the first variable, and the trivial connection with respect to the second. Similarly, $(\nabla^d)^*$ induces a metric connection ∇_2^d on $P_2^\star((\mathcal{E} \otimes \mathcal{L}^d)^*)$, and $\nabla_1^d \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_2^d$ is a metric connection on $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$.

We denote by ∂_x (respectively, ∂_y) the partial derivative of $\nabla_1^d \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_2^d$ with respect to the first (respectively, second) variable. Let

$$\partial_y^\sharp E_d(x, y) \in T_y\mathcal{X} \otimes (\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathcal{E} \otimes \mathcal{L}^d)_y^*$$

be defined by

$$\forall\, w \in T_y\mathcal{X}, \quad \partial_y^\sharp E_d(x, y) \cdot w^* = \partial_y E_d(x, y) \cdot w.$$

Similarly, let $\partial_x\, \partial_y^\sharp E_d(x, y) \in T_x^*\mathcal{X} \otimes T_y\mathcal{X} \otimes (\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathcal{E} \otimes \mathcal{L}^d)_y^*$ be defined by

$$\begin{aligned} &\forall\, (v, w) \in T_x\mathcal{X} \times T_y\mathcal{X}, \\ &\qquad \partial_x\, \partial_y^\sharp E_d(x, y) \cdot (v, w^*) = \partial_x\, \partial_y E_d(x, y) \cdot (v, w). \end{aligned}$$

The following corollary was proved in [22, Corollary 2.13].

Corollary 2.7. *Let $d \in \mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$, and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, for all x and $y \in \mathcal{X}$, we have*

$$\begin{aligned} \mathrm{Cov}(\nabla_x^d s, s(y)) &= \mathbb{E}[\nabla_x^d s \otimes s(y)^*] = \partial_x E_d(x, y), \\ \mathrm{Cov}(s(x), \nabla_y^d s) &= \mathbb{E}[s(x) \otimes (\nabla_y^d s)^*] = \partial_y^\sharp E_d(x, y), \\ \mathrm{Cov}(\nabla_x^d s, \nabla_y^d s) &= \mathbb{E}[\nabla_x^d s \otimes (\nabla_y^d s)^*] = \partial_x\, \partial_y^\sharp E_d(x, y). \end{aligned}$$

3. Estimates for the Bergman Kernel

In this section, we recall useful estimates for the Bergman kernels. First, in Subsection 3.1 we recall the definition of a preferred trivialization of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. Then, we state near-diagonal and off-diagonal estimates for a scaled version of E_d in Subsections 3.2 and 3.3.

3.1. Real normal trivialization.

Notation 3.1. In the following, $B_A(a, R)$ denotes the open ball of center a and radius R in the metric space A.

Let $d \in \mathbb{N}$, and let us define a preferred trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ in a neighborhood of any point of M. The properties of this trivialization were studied in Section 3.1 of [22]. Recall that the metric g on $\mathcal{X}$ is induced by the curvature of $(\mathcal{L}, h_{\mathcal{L}})$. Since $h_{\mathcal{L}}$ is compatible with the real structures, $c_{\mathcal{X}}$ is an isometry of $(\mathcal{X}, g)$ (see [22, Section 2.1] for details).

Let $R > 0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. Let also $x \in M$; then, the exponential map $\exp_x : B_{T_x\mathcal{X}}(0, 2R) \to B_{\mathcal{X}}(x, 2R)$ defines a chart around x such that

$$d_x c_{\mathcal{X}} = (\exp_x)^{-1} \circ c_{\mathcal{X}} \circ \exp_x .$$

In particular, since $T_xM = \ker(d_x c_{\mathcal{X}} - \mathrm{Id})$, we have that $\exp_x$ induces a diffeomorphism from $B_{T_xM}(0, 2R)$ to $B_M(x, 2R)$ that coincides with the exponential map of (M, g) at x.

We can now trivialize $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x, 2R)$, by identifying each fiber with $(\mathcal{E} \otimes \mathcal{L}^d)_x$ by parallel transport along geodesics, with respect to the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$. This defines a bundle map

$$\varphi_x : B_{T_x\mathcal{X}}(0, 2R) \times (\mathcal{E} \otimes \mathcal{L}^d)_x \to (\mathcal{E} \otimes \mathcal{L}^d)_{/B_{\mathcal{X}}(x,2R)}$$

that covers $\exp_x$. We call this trivialization the *real normal trivialization* of $\mathcal{E} \otimes \mathcal{L}^d$ around x.

Definition 3.2. A connection ∇^d on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ is said to be *real* if for every smooth section s we have

$$\forall\, y \in \mathcal{X}, \quad \nabla^d_y(c_d \circ s \circ c_{\mathcal{X}}) = c_d \circ \nabla^d_{c_{\mathcal{X}}(y)} s \circ d_y c_{\mathcal{X}}.$$

Such a connection induces a connection on $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$ by restriction.

Recall that c_d denotes the real structure of $\mathcal{E} \otimes \mathcal{L}^d$. Let $c_{d,x}$ denote its restriction to $(\mathcal{E} \otimes \mathcal{L}^d)_x$; then, we have that $(d_x c_{\mathcal{X}}, c_{d,x})$ is a $\mathbb{C}$-anti-linear involution of $B_{T_x\mathcal{X}}(0, 2R) \times (\mathcal{E} \otimes \mathcal{L}^d)_x$ that is compatible with the real structure on the first factor. Since the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$ is real (see [22, Lemma 3.4]), the real normal trivialization is well behaved with respect to the real structures, in the sense that for all $z \in B_{T_x\mathcal{X}}(0, 2R)$ and $\zeta \in (\mathcal{E} \otimes \mathcal{L}^d)_x$,

$$c_d(\varphi_x(z, \zeta)) = \varphi_x(d_x c_{\mathcal{X}} \cdot z, c_{d,x}(\zeta)).$$

Thus, φ_x can be restricted to a bundle map

$$B_{T_xM}(0, 2R) \times \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \to \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_{/B_M(x,2R)}$$

that covers $\exp_x$.

Finally, it is known (cf. [22, Section 3.1]) that φ_x is a unitary trivialization (i.e., its restriction to each fiber is an isometry). Similarly, its restriction to the real locus is an orthogonal trivialization of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_{/B_M(x,2R)}$.

The point is the following. The usual differentiation for maps from $T_x\mathcal{X}$ to $(\mathcal{E}\otimes\mathcal{L}^d)_x$ defines locally a connection $\tilde{\nabla}^d$ on $(\mathcal{E}\otimes\mathcal{L}^d)_{/B_{\mathcal{X}}(x,2R)}$ via the real normal trivialization. Since this trivialization is well behaved with respect to both the real and the metric structures, $\tilde{\nabla}^d$ is a real metric connection. Then, by a partition of unity argument, there exists a global real metric connection ∇^d on $\mathcal{E} \otimes \mathcal{L}^d$ that agrees with $\tilde{\nabla}^d$ on $B_{\mathcal{X}}(x, R)$; that is, ∇^d is trivial in the real normal trivialization, in a neighborhood of x. The existence of such a connection will be useful in the proof of our main theorem.

3.2. Near-diagonal estimates. In this section, we state estimates for a scaled version of the Bergman kernel in a neighborhood of the diagonal of $M \times M$. As in the previous section, let $R > 0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. Let $x \in M$; then, the real normal trivialization φ_x induces a trivialization of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ over $B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R)$ that covers $\exp_x \times \exp_x$. This trivialization agrees with the real normal trivialization of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ around (x, x).

In the normal chart $\exp_x$, we note that the Riemannian measure $\mathrm{d}V_{\mathcal{X}}$ admits a positive density with respect to the Lebesgue measure of $(T_x\mathcal{X}, g_x)$, denoted by $\kappa : B_{T_x\mathcal{X}}(0, 2R) \to \mathbb{R}_+$. Then, in the chart $\exp_x : B_{T_xM}(0, 2R) \to B_M(x, 2R)$, the density of $|\mathrm{d}V_M|$ with respect to the Lebesgue measure of (T_xM, g_x) is $\sqrt{\kappa}$.

Let us identify E_d with its expression in the real normal trivialization of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ around (x, x). Thus, the restriction of E_d to the real locus is a map from $T_xM \times T_xM$ to $\mathrm{End}(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)$. Then, by [10, Theorem 4.18'] (see also [22, Theorem 3.5] for a statement with the same notation as the present paper), we get the following estimate for E_d and its derivatives of order at most 6.

Theorem 3.3 (Dai-Liu-Ma). *There exist C_1 and $C_2 > 0$, such that, for all $k \in \{0, 1, \dots, 6\}$, all $d \in \mathbb{N}^*$, all $x \in M$, and all $w, z \in B_{T_xM}(0, R)$,*

$$\begin{aligned}&\left\|\mathrm{D}^k_{(w,z)}\left(E_d(w, z) - \left(\frac{d}{\pi}\right)^n \frac{\exp(-(d/2)\|z - w\|^2)}{\sqrt{\kappa(w)}\sqrt{\kappa(z)}} \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)\right\| \\ &\leqslant C_1 d^{n+k/2-1}(1 + \sqrt{d}(\|w\| + \|z\|))^{2n+12} \exp(-C_2\sqrt{d}\|z - w\|) + O(d^{-\infty}),\end{aligned}$$

where D^k *is the* k*-th differential for a map from* $T_xM \times T_xM$ *to* $\mathrm{End}(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)$*, the norm on* T_xM *is induced by* g_x*, and the norm on* $(T_x^*M)^{\otimes k} \otimes \mathrm{End}((\mathcal{E} \otimes \mathcal{L}^d)_x)$ *is induced by* g_x *and* $(h_d)_x$*.*

The notation $O(d^{-\infty})$ *means that, for any* $\ell \in \mathbb{N}$*, this term is* $O(d^{-\ell})$ *with a constant that does not depend on* x*,* w*,* z*, nor* d*.*

Recall that x is fixed. We denote by e_d the following scaled version of the Bergman kernel around x:

$$\forall\, w, z \in B_{T_xM}(0, 2R\sqrt{d}), \quad e_d(w,z) = \left(\frac{\pi}{d}\right)^n E_d\left(\exp_x\left(\frac{w}{\sqrt{d}}\right), \exp_x\left(\frac{z}{\sqrt{d}}\right)\right). \tag{3.1}$$

We consider e_d as a map with values in $\mathrm{End}(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)$ using the real normal trivialization around x. Note that e_d highly depends on x, even if this is not reflected in the notation. In the following, the base point x will always be clear from the context.

Let $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $\xi(w,z) = \exp(-\frac{1}{2}\|w - z\|^2)$, where $\|\cdot\|$ is the usual Euclidean norm. Let $x \in M$; then, any isometry from T_xM to $\mathbb{R}^n$ allows us to see ξ as a map from $T_xM \times T_xM$ to $\mathbb{R}$. Let b_n be a positive constant depending only on n and whose value will be chosen later on. Then, using the same notation as in Theorem 3.3 we get the following.

Proposition 3.4. *Let* $\alpha \in (0,1)$*; then, there exists* $C > 0$*, depending only on* α*,* n*, and the geometry of* $\mathcal{X}$*, such that* $\forall\, k \in \{0, 1, \dots, 6\}$*,* $\forall\, d \in \mathbb{N}^*$*,* $\forall\, x \in M$*,* $\forall\, w, z \in B_{T_xM}(0, b_n \ln d)$*. We have*

$$\|\mathrm{D}^k_{(w,z)} e_d - (\mathrm{D}^k_{(w,z)}\xi)\, \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\| \leqslant C d^{-\alpha}.$$

Proof. First, we apply Theorem 3.3 for the scaled kernel e_d. This yields that, for all $k \in \{0, 1, \dots, 6\}$, all $d \in \mathbb{N}^*$, all $x \in M$, and all $w, z \in B_{T_xM}(0, b_n \ln d)$,

$$\begin{aligned} &\left\| \mathrm{D}^k_{(w,z)}\left(e_d(w,z) - \frac{\xi(w,z)}{\sqrt{\tilde{\kappa}(w)\tilde{\kappa}(z)}}\, \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)\right\| \\ &\qquad \leqslant \frac{C_1}{d}(1 + 2b_n \ln d)^{2n+12} + O(d^{-\infty}) = O(d^{-\alpha}), \end{aligned}$$

where $\tilde{\kappa} : z \mapsto \kappa(z/\sqrt{d})$. Since we used normal coordinates to define κ, the following relations hold uniformly on $B_{T_xX}(0, R)$:

$$\kappa(z) = 1 + O(\|z\|^2), \quad \mathrm{D}_z\kappa = O(\|z\|) \quad \text{and} \quad \forall\, k \in \{2, \dots, 6\},\ \mathrm{D}^k_z\kappa = O(1).$$

By compactness, these estimates can be made independent of $x \in M$. Then, we obtain the following estimates for $\tilde{\kappa}$ and its derivatives, uniformly in $x \in M$

and $z \in B_{T_xM}(0, b_n \ln d)$:

$$\tilde{\kappa}(z) = 1 + O\left(\frac{(b_n \ln d)^2}{d}\right), \quad \mathrm{D}_z\tilde{\kappa} = O\left(\frac{b_n \ln d}{d}\right)$$
$$\text{and} \quad \forall\, k \in \{2, \dots, 6\},\ \mathrm{D}_z^k\tilde{\kappa} = O\left(\frac{1}{d}\right).$$

Therefore, for all $k \in \{0, 1, \dots, 6\}$, for all $d \in \mathbb{N}^*$, for all $x \in M$, and for all $w, z \in B_{T_xM}(0, b_n \ln d)$,

$$\left\| \mathrm{D}^k_{(w,z)}\left(\frac{\xi(w,z)}{\sqrt{\tilde{\kappa}(w)\tilde{\kappa}(z)}}\right) - \mathrm{D}^k_{(w,z)}\xi \right\| = O(d^{-\alpha}). \qquad \square$$

We will use the expressions of some of the partial derivatives of ξ. Let us choose any orthonormal basis of T_xM and denote by ∂_{x_i} (respectively, ∂_{y_i}) the partial derivative with respect to the i-th component of the first (respectively, second) variable.

Lemma 3.5. *Let $i, j \in \{1, \dots, n\}$.*
For all $w = (w_1, \dots, w_n)$ and $z = (z_1, \dots, z_n) \in T_xM$, we have

$$\partial_{x_i}\xi(w,z) = -(w_i - z_i)\exp\left(-\frac{1}{2}\|w - z\|^2\right),$$
$$\partial_{y_j}\xi(x,y) = (w_j - z_j)\exp\left(-\frac{1}{2}\|w - z\|^2\right),$$

and

$$\partial_{x_i}\partial_{y_j}\xi(x,y) = (\delta_{ij} - (w_i - z_i)(w_j - z_j))\exp\left(-\frac{1}{2}\|w - z\|^2\right),$$

where δ_{ij} equals 1 if $i = j$ and 0 otherwise.

Proof. This is given by a direct computation. $\square$

3.3. Off-diagonal estimates. Finally, let us recall estimates quantifying the long range decay of the Bergman kernel E_d. These estimates were proved by Ma and Marinescu in [24, Theorem 5].

Let S be a smooth section of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)^*$, and let $x, y \in M$. Denote by $\|S(x,y)\|_{C^k}$ the maximum of the norms of S and its derivatives of order at most k at $(x,y) \in M \times M$, where the derivatives of S are computed with respect to the connection induced by the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$ and the Levi-Civita connection on M. The norms are the natural ones induced by h_d and g.

Theorem 3.6 (Ma-Marinescu). *There exist $d_0 \in \mathbb{N}^*$, and positive constants C_1 and C_2 such that, for all $k \in \{0, 1, 2\}$, all $d \geqslant d_0$, and all $x, y \in M$, we have*

$$\|E_d(x,y)\|_{C^k} \leqslant C_1 d^{n+k/2}\exp(-C_2\sqrt{d}\rho_g(x,y)),$$

where $\rho_g(\cdot,\cdot)$ denotes the geodesic distance in (M, g).

4. Properties of the Limit Distribution

The estimates of Section 3.2 show that the family $(s_d(x))_{x\in M}$ of random fields has a local scaling limit around any point $x \in M$, as d goes to infinity. Moreover, the limit field does not depend on x. The limit is a Gaussian centered random process from $\mathbb{R}^n$ to $\mathbb{R}^r$ whose correlation kernel is $e_\infty : (w,z) \mapsto \xi(w,z)I_r$, where I_r stands for the identity of $\mathbb{R}^r$ and ξ was defined in Section 3.2. This limit process is known as the Bargmann-Fock process.

The goal of this section is to establish some properties of the Bargmann-Fock process. These results will be useful in the next section to prove that, for d large enough, the local behavior of s_d around any given $x \in M$ is the same as that of the limit process.

In the following, we denote by $(s(z))_{z\in\mathbb{R}^n}$ a copy of the Bargmann-Fock process. Since ξ is smooth, we can assume the trajectories of s are smooth. Note that s is both stationary and isotropic. Moreover, since $e_\infty = \xi I_r$, the field s is just a tuple of r independent identically distributed centered Gaussian fields whose correlation kernel is ξ.

4.1. Variance of the values. The first thing we want to understand about s is the distribution of $(s(0), s(z)) \in \mathbb{R}^r \oplus \mathbb{R}^r$ for any $z \in \mathbb{R}^n$. In the following, we canonically identify $\mathbb{R}^r \oplus \mathbb{R}^r$ with $\mathbb{R}^2 \otimes \mathbb{R}^r$.

Let $z \in \mathbb{R}^n$; then, $(s(0), s(z))$ is a centered Gaussian vector in $\mathbb{R}^2 \otimes \mathbb{R}^r$ with variance operator

$$\Theta(z) = \begin{pmatrix} e_\infty(0,0) & e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z) \end{pmatrix} = \begin{pmatrix} \xi(0,0)I_r & \xi(0,z)I_r \\ \xi(z,0)I_r & \xi(z,z)I_r \end{pmatrix} = \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix} \otimes I_r. \tag{4.1}$$

Let $Q = (1/\sqrt{2}) \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in O_2(\mathbb{R})$ denote the rotation of angle $\pi/4$ in $\mathbb{R}^2$. We can explicitly diagonalize $\Theta(z)$ as follows.

Lemma 4.1. *For any $z \in \mathbb{R}^n$ we have the following:*

$$(Q \otimes I_r)\Theta(z)(Q \otimes I_r)^{-1} = \begin{pmatrix} 1 - e^{-(1/2)\|z\|^2} & 0 \\ 0 & 1 + e^{-(1/2)\|z\|^2} \end{pmatrix} \otimes I_r.$$

Proof. Since $(Q \otimes I_r)^{-1} = Q^{\mathrm{t}} \otimes I_r$, by equation (4.1), it is enough to notice that

$$Q \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix} Q^{\mathrm{t}} = \begin{pmatrix} 1 - e^{-(1/2)\|z\|^2} & 0 \\ 0 & 1 + e^{-(1/2)\|z\|^2} \end{pmatrix}. \qquad \square$$

Lemma 4.2. *For all $z \in \mathbb{R}^n$, $\det(\Theta(z)) = (1 - e^{-\|z\|^2})^r$. In particular, the distribution of $(s(0), s(z))$ is non-degenerate for all $z \in \mathbb{R}^n \setminus \{0\}$.*

Proof. We take the determinant of both sides in equation (4.1). $\square$

4.2. Variance of the* 1*-jets. Let us now study the variance structure of the 1-jets of s. For any $z \in \mathbb{R}^n$, we know that $(s(0), s(z), d_0 s, d_z s)$ is a centered Gaussian vector in

$$\mathbb{R}^r \oplus \mathbb{R}^r \oplus ((\mathbb{R}^n)^* \otimes \mathbb{R}^r) \oplus ((\mathbb{R}^n)^* \otimes \mathbb{R}^r) \simeq (\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r.$$

Our goal in this section is to better understand its variance operator $\Omega(z)$. In the following, we write $\Omega(z)$ by blocks according to the previous splitting. Let ∂_x (respectively, ∂_y) denote the partial derivative with respect to the first (respectively, second) variable for maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathrm{End}(\mathbb{R}^r)$. Let us also define $\partial_y^\sharp$ as in Subsection 2.4. Then, we have

$$\begin{aligned}\Omega(z) &= \left(\begin{array}{cc|cc} e_\infty(0,0) & e_\infty(0,z) & \partial_y^\sharp e_\infty(0,0) & \partial_y^\sharp e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z) & \partial_y^\sharp e_\infty(z,0) & \partial_y^\sharp e_\infty(z,z) \\ \hline \partial_x e_\infty(0,0) & \partial_x e_\infty(0,z) & \partial_x \partial_y^\sharp e_\infty(0,0) & \partial_x \partial_y^\sharp e_\infty(0,z) \\ \partial_x e_\infty(z,0) & \partial_x e_\infty(z,z) & \partial_x \partial_y^\sharp e_\infty(z,0) & \partial_x \partial_y^\sharp e_\infty(z,z) \end{array}\right) \\ &= \Omega'(z) \otimes I_r, \end{aligned} \tag{4.2}$$

where

$$\Omega'(z) = \left(\begin{array}{cc|cc} \xi(0,0) & \xi(0,z) & \partial_y^\sharp \xi(0,0) & \partial_y^\sharp \xi(0,z) \\ \xi(z,0) & \xi(z,z) & \partial_y^\sharp \xi(z,0) & \partial_y^\sharp \xi(z,z) \\ \hline \partial_x \xi(0,0) & \partial_x \xi(0,z) & \partial_x \partial_y^\sharp \xi(0,0) & \partial_x \partial_y^\sharp \xi(0,z) \\ \partial_x \xi(z,0) & \partial_x \xi(z,z) & \partial_x \partial_y^\sharp \xi(z,0) & \partial_x \partial_y^\sharp \xi(z,z) \end{array}\right).$$

Let $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ be any orthonormal basis of $\mathbb{R}^n$ such that we have $z = \|z\| \, \partial/\partial x_1$, and let $(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ denote its dual basis. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$; we denote by $\mathcal{B}_z$ the following orthonormal basis of $\mathbb{R}^2 \otimes (\mathbb{R} \oplus (\mathbb{R}^n)^*) \simeq \mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$:

$$\mathcal{B}_z = (e_1 \otimes 1, e_2 \otimes 1, e_1 \otimes \mathrm{d}x_1, e_2 \otimes \mathrm{d}x_1, \dots, e_1 \otimes \mathrm{d}x_n, e_2 \otimes \mathrm{d}x_n).$$

Lemma 4.3. *For any $z \in \mathbb{R}^n$, the matrix of $\Omega'(z)$ in the basis $\mathcal{B}_z$ is*

$$\left(\begin{array}{c|c} \tilde{\Omega}(\|z\|^2) & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right),$$

where I_{n-1} is the identity matrix of size $n-1$ and, for all $t \geqslant 0$, we set

$$\tilde{\Omega}(t) = \begin{pmatrix} 1 & e^{-(1/2)t} & 0 & -\sqrt{t}e^{-(1/2)t} \\ e^{-(1/2)t} & 1 & \sqrt{t}e^{-(1/2)t} & 0 \\ 0 & \sqrt{t}e^{-(1/2)t} & 1 & (1-t)e^{-(1/2)t} \\ -\sqrt{t}e^{-(1/2)t} & 0 & (1-t)e^{-(1/2)t} & 1 \end{pmatrix}. \tag{4.3}$$

Proof. A direct computation, by using the fact that $z = (\|z\|, 0, \dots, 0)$ in the basis $(\partial/\partial x_1, \dots, \partial/\partial x_n)$, yields the result. Recall that the partial derivatives of ξ are given by Lemma 3.5. ∎

Let $z \in \mathbb{R}^n$, and recall that $z^* \in (\mathbb{R}^n)^*$ was defined as $z^* = \langle \cdot, z\rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical scalar product of $\mathbb{R}^n$. We denote by $z^* \otimes z \in \mathrm{End}((\mathbb{R}^n)^*)$ the map $\eta \mapsto \eta(z) z^*$. Then, from Lemma 4.3 we see that, as an operator on $\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$,

$$\Omega'(z) = \left(\begin{array}{cc|cc} 1 & e^{-(1/2)\|z\|^2} & 0 & -e^{-(1/2)\|z\|^2} z \\ e^{-(1/2)\|z\|^2} & 1 & e^{-(1/2)\|z\|^2} z & 0 \\ \hline 0 & e^{-(1/2)\|z\|^2} z^* & I_n & e^{-(1/2)\|z\|^2}(I_n - z^* \otimes z) \\ -e^{-(1/2)\|z\|^2} z^* & 0 & e^{-(1/2)\|z\|^2}(I_n - z^* \otimes z) & I_n \end{array}\right), \tag{4.4}$$

where z^* is to be understood as the constant map $t \mapsto z^*$ from $\mathbb{R}$ to $(\mathbb{R}^n)^*$, z is to be understood as the evaluation at the point z from $(\mathbb{R}^n)^*$ to $\mathbb{R}$, and I_n is the identity of $\mathbb{R}^n$. Indeed, both sides of (4.4) have the same matrix in the basis $\mathcal{B}_z$.

We will now diagonalize explicitly $\Omega(z)$, as we did for $\Theta(z)$ in the previous section. The main step is to diagonalize $\tilde{\Omega}(z)$.

Definitions 4.4. We denote by v_1, v_2, v_3, v_4, and a the following functions from $[0, +\infty)$ to $\mathbb{R}$:

$$v_1 : t \longmapsto 1 - e^{-(1/2)t}\left(\frac{t}{2} - \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$v_2 : t \longmapsto 1 - e^{-(1/2)t}\left(\frac{t}{2} + \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$v_3 : t \longmapsto 1 + e^{-(1/2)t}\left(\frac{t}{2} - \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$v_4 : t \longmapsto 1 + e^{-(1/2)t}\left(\frac{t}{2} + \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$a : t \longmapsto \frac{1 - \frac{t}{2}}{\sqrt{1 + \left(\frac{t}{2}\right)^2}}.$$

Note that, for all $t \geqslant 0$, $|a(t)| \leqslant 1$, so that the following makes sense.

Definitions 4.5. Let $b_+ : t \mapsto \sqrt{1+a(t)}$ and $b_- : t \mapsto \sqrt{1-a(t)}$ from $[0,+\infty)$ to $\mathbb{R}$. For all $t \geqslant 0$, let us also denote

$$P(t) = \frac{1}{2}\begin{pmatrix} b_-(t) & -b_-(t) & -b_+(t) & -b_+(t) \\ b_+(t) & -b_+(t) & b_-(t) & b_-(t) \\ b_-(t) & b_-(t) & -b_+(t) & b_+(t) \\ b_+(t) & b_+(t) & b_-(t) & -b_-(t) \end{pmatrix}.$$

One can check that, for all $t \geqslant 0$, $P(t)$ is an orthogonal matrix.

Lemma 4.6. *For every* $t \in [0,+\infty)$*, we have*

$$P(t)\tilde{\Omega}(t)P(t)^{-1} = \begin{pmatrix} v_1(t) & 0 & 0 & 0 \\ 0 & v_2(t) & 0 & 0 \\ 0 & 0 & v_3(t) & 0 \\ 0 & 0 & 0 & v_4(t) \end{pmatrix}.$$

Proof. See Appendix A. ◻

Corollary 4.7. *Let* $z \in \mathbb{R}^n$*. Identifying* $\Omega'(z)$ *with its matrix in* $\mathcal{B}_z$*, we have*

$$\left(\begin{array}{c|c} P(\|z\|^2) & 0 \\ \hline 0 & Q \otimes I_{n-1} \end{array}\right) \Omega'(z) \left(\begin{array}{c|c} P(\|z\|^2) & 0 \\ \hline 0 & Q \otimes I_{n-1} \end{array}\right)^{-1}$$
$$= \left(\begin{array}{cccc|c} v_1(\|z\|^2) & 0 & 0 & 0 & \\ 0 & v_2(\|z\|^2) & 0 & 0 & \\ 0 & 0 & v_3(\|z\|^2) & 0 & 0 \\ 0 & 0 & 0 & v_4(\|z\|^2) & \\ \hline & & 0 & & \begin{pmatrix} 1-e^{-(1/2)\|z\|^2} & 0 \\ 0 & 1+e^{-(1/2)\|z\|^2} \end{pmatrix} \otimes I_{n-1} \end{array}\right).$$

By equation (4.2), we get a diagonalization of $\Omega(z)$ by tensoring each factor by I_r in Corollary 4.7.

Lemma 4.8. *For all* $z \in \mathbb{R}^n \setminus \{0\}$*, we have* $\det(\Omega(z)) > 0$*. That is, the distribution of* $(s(0), s(z), d_0 s, d_z s)$ *is non-degenerate.*

Proof. See Appendix A. ◻

4.3. Conditional variance of the derivatives. The next step is to study the conditional distribution of $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$, for any $z \in \mathbb{R}^n \setminus \{0\}$. Recall that $(s(0), s(z), d_0 s, d_z s)$ is a centered Gaussian vector with variance $\Omega(z)$ (see equation (4.2)). Moreover, if $z \neq 0$, the distribution of $(s(0), s(z))$ is non-degenerate by Lemma 4.2.

Thus, $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$ is a centered Gaussian vector in $((\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r$ with variance operator

$$\begin{aligned}\Lambda(z) = &\begin{pmatrix} \partial_x \partial_y^\sharp e_\infty(0,0) & \partial_x \partial_y^\sharp e_\infty(0,z) \\ \partial_x \partial_y^\sharp e_\infty(z,0) & \partial_x \partial_y^\sharp e_\infty(z,z)\end{pmatrix} \\ &- \begin{pmatrix} \partial_x e_\infty(0,0) & \partial_x e_\infty(0,z) \\ \partial_x e_\infty(z,0) & \partial_x e_\infty(z,z)\end{pmatrix} \begin{pmatrix} e_\infty(0,0) & e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z)\end{pmatrix}^{-1} \\ &\times \begin{pmatrix} \partial_y^\sharp e_\infty(0,0) & \partial_y^\sharp e_\infty(0,z) \\ \partial_y^\sharp e_\infty(z,0) & \partial_y^\sharp e_\infty(z,z)\end{pmatrix}.\end{aligned}$$

By (4.2) and (4.4), for all $z \in \mathbb{R}^n \setminus \{0\}$, we have $\Lambda(z) = \Lambda'(z) \otimes I_r$, where

$$\begin{aligned}\Lambda'(z) &\\ = &\begin{pmatrix} I_n - \dfrac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-(1/2)\|z\|^2}\left(I_n - \dfrac{1}{1-e^{-\|z\|^2}} z^* \otimes z\right) \\ e^{-(1/2)\|z\|^2}\left(I_n - \dfrac{1}{1-e^{-\|z\|^2}} z^* \otimes z\right) & I_n - \dfrac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix}\end{aligned} \tag{4.5}$$

As in the previous section, let us denote by $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ an orthonormal basis of $\mathbb{R}^n$ such that $z = \|z\|\, \partial/\partial x_1$, and let $(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ denote its dual basis. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$. We define $\mathcal{B}'_z$ to be the following orthonormal basis of $\mathbb{R}^2 \otimes (\mathbb{R}^n)^* \simeq (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$:

$$\mathcal{B}'_z = (e_1 \otimes \mathrm{d}x_1, e_2 \otimes \mathrm{d}x_1, \dots, e_1 \otimes \mathrm{d}x_n, e_2 \otimes \mathrm{d}x_n).$$

Lemma 4.9. *For any $z \in \mathbb{R}^n \setminus \{0\}$, the matrix of $\Lambda'(z)$ in the basis $\mathcal{B}'_z$ is*

$$\left(\begin{array}{c|c} \tilde{\Lambda}(\|z\|^2) & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right),$$

where, for all $t > 0$, we set

$$\tilde{\Lambda}(t) = \begin{pmatrix} 1 - \dfrac{te^{-t}}{1-e^{-t}} & e^{-(1/2)t}\left(1 - \dfrac{t}{1-e^{-t}}\right) \\ e^{-(1/2)t}\left(1 - \dfrac{t}{1-e^{-t}}\right) & 1 - \dfrac{te^{-t}}{1-e^{-t}} \end{pmatrix}.$$

Proof. Since $z = \|z\|\, \partial/\partial x_1$, we have $z^* \otimes z = \|z\|^2\, \mathrm{d}x_1 \otimes \partial/\partial x_1$. Hence, the matrix of $z^* \otimes z$ in $(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ is

$$\left(\begin{array}{c|c} \|z\|^2 & 0 \\ \hline 0 & 0 \end{array}\right).$$

Then, the conclusion follows from equation (4.5). ◻

Remark 4.10. We can extend continuously $\tilde{\Lambda}$ at $t = 0$ by setting $\tilde{\Lambda}(0) = 0$. This yields continuous extensions of Λ' and Λ at $z = 0$. Note that $\Lambda(0)$ is not the variance operator of $(d_0 s, d_0 s)$ given that $s(0) = 0$.

Definitions 4.11. Let u_1, u_2 denote the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$u_1 : t \longmapsto \frac{1 - e^{-t} + te^{-(1/2)t}}{1 + e^{-(1/2)t}}, \quad u_2 : t \longmapsto \begin{cases} \dfrac{1 - e^{-t} - te^{-(1/2)t}}{1 - e^{-(1/2)t}} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Once again, we will need an explicit diagonalization of $\Lambda(z)$. Such a diagonalization is given by the following lemma, once we tensor each factor by I_r.

Lemma 4.12. *Let $z \in \mathbb{R}^n$. Identifying $\Lambda'(z)$ with its matrix in $\mathcal{B}'_z$, we have*

$$(Q \otimes I_n)\Lambda'(z)(Q \otimes I_n)^{-1} = \left(\begin{array}{cc|c} u_1(\|z\|^2) & 0 & \\ 0 & u_2(\|z\|^2) & 0 \\ \hline & 0 & \begin{pmatrix} 1 - e^{-(1/2)\|z\|^2} & 0 \\ 0 & 1 + e^{-(1/2)\|z\|^2} \end{pmatrix} \otimes I_{n-1} \end{array}\right).$$

Proof. By Lemma 4.9, we only need to check that, for all $t \geqslant 0$,

$$Q\tilde{\Lambda}(t)Q^{\mathrm{t}} = \begin{pmatrix} u_1(t) & 0 \\ 0 & u_2(t) \end{pmatrix}. \qquad \square$$

Lemma 4.13. *For all $z \in \mathbb{R}^n \setminus \{0\}$, we have $\det(\Lambda(z)) > 0$, that is, the distribution of $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$ is non-degenerate.*

Proof. See Appendix A. $\square$

By Lemma 4.8, $\Omega(z)$ is a positive self-adjoint operator on

$$(\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r,$$

for all $z \in \mathbb{R}^n \setminus \{0\}$, and so is its inverse. Hence, $\Omega(z)^{-1}$ admits a unique positive square root, which we denote by $\Omega(z)^{-1/2}$. Similarly, by Lemma 4.13, $\Lambda(z)$ is a positive self-adjoint operator on $((\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r$, and we denote by $\Lambda(z)^{1/2}$ its positive square root.

Lemma 4.14. *The map $z \longmapsto (0\ \Lambda(z)^{1/2})\Omega(z)^{-1/2}$ is bounded on $\mathbb{R}^n \setminus \{0\}$.*

Proof. See Appendix A. $\square$

4.4. Finiteness of the leading constant. The goal of this section is to prove that the constant $\mathcal{I}_{n,r}$ defined by equation (1.2) and appearing in Theorem 1.6 is well defined and finite.

Definition 4.15. Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$. For every $t > 0$ we set

$$\mathrm{D}_{n,r}(t) = \frac{\mathbb{E}[|\det^{\perp}(X(t))|\,|\det^{\perp}(Y(t))|]}{(1-e^{-t})^{r/2}} - (2\pi)^r \left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2,$$

where $(X(t), Y(t))$ is the centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ defined in Definition 1.5.

By the definition of $\mathcal{I}_{n,r}$ (see equation (1.2)), we have

$$\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} \mathrm{D}_{n,r}(t) t^{(n-2)/2}\,\mathrm{d}t.$$

Hence, we must prove $t \mapsto \mathrm{D}_{n,r}(t)t^{(n-2)/2}$ is integrable on $(0, +\infty)$, which boils down to computing the asymptotic expansions of $\mathbb{E}[|\det^{\perp}(X(t))|\,|\det^{\perp}(Y(t))|]$ as $t \to 0$ and as $t \to +\infty$.

Let us now relate this to the Bargmann-Fock process $(s(z))_{z\in\mathbb{R}^n}$.

Lemma 4.16. *Let $z \in \mathbb{R}^n \setminus \{0\}$. Let $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ be an orthonormal basis of $\mathbb{R}^n$ such that $z = \|z\|\,\partial/\partial x_1$, and let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{R}^r$. Then, the matrices of $d_0 s$ and $d_z s$ in these bases, given that $s(0) = 0 = s(z)$, form a random vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ which is distributed as $(X(\|z\|^2), Y(\|z\|^2))$.*

Proof. Let us denote by $\tilde{X}(z)$ and $\tilde{Y}(z)$ the matrices of $d_0 s$ and $d_z s$ in the bases $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $(\zeta_1, \dots, \zeta_r)$, given that $s(0) = 0 = s(z)$. We denote by $\tilde{X}_{ij}(z)$ (respectively, $\tilde{Y}_{ij}(z)$) the coefficients of $\tilde{X}$ (respectively, $\tilde{Y}$) for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$. By Lemma 4.9, the couples $(\tilde{X}_{ij}, \tilde{Y}_{ij})$ are centered Gaussian vectors in $\mathbb{R}^2$ which are independant from one another. Moreover, the variance matrix of $(\tilde{X}_{ij}(z), \tilde{Y}_{ij}(z))$ equals

$$\tilde{\Lambda}(\|z\|^2) \qquad \text{if } j = 1,$$

and

$$\begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix} \quad \text{otherwise.}$$

By Definition 1.5, this is precisely saying that $(\tilde{X}(z), \tilde{Y}(z))$ is distributed as $(X(\|z\|^2), Y(\|z\|^2))$. ∎

Lemma 4.17. *Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$. Then, as $t \to 0$, we have the following:*

$$\mathbb{E}[|\det^{\perp}(X(t))|\,|\det^{\perp}(Y(t))|] \sim \begin{cases} \dfrac{(n-1)!}{(n-r-1)!} & \text{if } r < n, \\ \dfrac{n!}{2}t & \text{if } r = n. \end{cases}$$

Proof. See Appendix A. □

Lemma 4.18. *For all $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, we have the following as $t \to +\infty$:*

$$\mathbb{E}[|\det^{\perp}(X(t))|\,|\det^{\perp}(Y(t))|] = (2\pi)^r \left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2 + O(te^{-t/2}).$$

Proof. See Appendix A. □

Lemmas 4.17 and 4.18 and the definition of $\mathrm{D}_{n,r}$ (Definition 4.15) allow us to derive the following.

Corollary 4.19. *Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$; then, we have*

$$t^{(n-2)/2}\mathrm{D}_{n,r}(t) = \begin{cases} O\left(\dfrac{1}{\sqrt{t}}\right) & \text{as } t \to 0, \\ O(e^{-t/4}) & \text{as } t \to +\infty. \end{cases}$$

In particular, $\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} \mathrm{D}_{n,r}(t)t^{(n-2)/2}\,\mathrm{d}t$ is well defined and finite.

5. Proof of Theorem 1.6

This section is concerned with the proof of our main result (Theorem 1.6). Recall that $\mathcal{X}$ is a compact Kähler manifold of complex dimension $n \geqslant 1$ defined over the reals, and that M denotes its real locus, assumed to be non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ real Hermitian vector bundle and $\mathcal{L} \to \mathcal{X}$ be a real Hermitian line bundle whose curvature form is ω, the Kähler form of $\mathcal{X}$. We assume that $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. For all $d \in \mathbb{N}$, we still denote by E_d the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Finally, let s_d denote a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, whose real zero set is denoted by Z_d, and let $|\mathrm{d}V_d|$ denote the Riemannian volume measure on Z_d.

In Subsection 5.1, we recall Kac-Rice formulas and use them to derive an integral expression of $\mathrm{Var}(|\mathrm{d}V_d|)$. Subsection 5.2 is concerned with the study of some relevant random variables related to $(s_d(x))_{x \in M}$. Finally, we conclude the proof in two steps, in Subsections 5.3 and 5.4.

5.1. Kac-Rice formulas In this section, we use Kac-Rice formulas to derive an integral expression of $\mathrm{Var}(|\mathrm{d}V_d|)$. Classical references for this material are [1, Chapter 11.5] and [3, Theorem 6.3]. Since our probability space is the finite-dimensional vector space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, it is possible to derive Kac-Rice formulas under weaker hypothesis than those given in [1] and [3]. This uses Federer's coaera formula and the so-called double fibration trick (see [21, Appendix C] and the references therein). The first Kac-Rice formula we state (Theorem 5.1 below) was proved in [21, Theorem 5.3], and the second (Theorem 5.5 below) was proved in [22, Theorem 4.4].

Recall that the Jacobian $|\det^{\perp}(L)|$ of an operator L was defined in Definition 1.4, that d_1 was defined in Lemma 2.3, and that a connection is said to be real if it satisfies the condition given in Definition 3.2.

Theorem 5.1 (Kac-Rice formula 1). *Let $d \geqslant d_1$, let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$, and let $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, for all $\phi \in C^0(M)$,*

$$\begin{aligned}(5.1)\quad \mathbb{E}\left[\int_{x \in Z_d} \phi(x)\,|\mathrm{d}V_d|\right] \\ = (2\pi)^{-r/2} \int_{x \in M} \frac{\phi(x)}{|\det^{\perp}(\mathrm{ev}_x^d)|} \mathbb{E}\left[|\det^{\perp}(\nabla_x^d s_d)| : s_d(x) = 0\right] |\mathrm{d}V_M|.\end{aligned}$$

The expectation on the righthand side of equation (5.1) is to be understood as the conditional expectation of $|\det^{\perp}(\nabla_x^d s_d)|$ given that $s_d(x) = 0$.

Notation 5.2. Let $\Delta = \{(x, y) \in M^2 \mid x = y\}$ denote the diagonal of M^2.

Definition 5.3. Let $d \in \mathbb{N}$ and let $(x, y) \in M^2 \setminus \Delta$; we denote by $\mathrm{ev}_{x,y}^d$ the evaluation map

$$\begin{aligned}\mathrm{ev}_{x,y}^d : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) &\longrightarrow \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_y, \\ s &\longmapsto (s(x), s(y)).\end{aligned}$$

Lemma 5.4. *There exists $d_2 \in \mathbb{N}$, such that for all $(x, y) \in M^2 \setminus \Delta$, $\mathrm{ev}_{x,y}^d$ is surjective.*

This was proved in [22, Proposition 4.2] in the case $r < n$, by using Kodaira's embedding theorem. The proof can be adapted verbatim to the case $r \leqslant n$. We will give an alternative proof using only estimates on the Bergman kernel (Lemmas 5.23 and 5.26) (see p. 1684 below).

Theorem 5.5 (Kac-Rice formula 2). *Let $d \geqslant d_2$, let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$, and let $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, we have that, for all ϕ_1 and $\phi_2 \in C^0(M)$,*

$$\begin{aligned}(5.2)\quad \mathbb{E}\left[\int_{(x,y) \in (Z_d)^2 \setminus \Delta} \phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right] \\ = \frac{1}{(2\pi)^r} \int_{(x,y) \in M^2 \setminus \Delta} \frac{\phi_1(x)\phi_2(y)}{|\det^{\perp}(\mathrm{ev}_{x,y}^d)|} \\ \times \mathbb{E}\left[|\det^{\perp}(\nabla_x^d s_d)|\,|\det^{\perp}(\nabla_y^d s_d)| : \mathrm{ev}_{x,y}^d(s_d) = 0\right] |\mathrm{d}V_M|^2.\end{aligned}$$

Here, $|\mathrm{d}V_M|^2$ (respectively, $|\mathrm{d}V_d|^2$) stands for the product measure on M^2 (respectively, $(Z_d)^2$) induced by $|\mathrm{d}V_M|$ (respectively $|\mathrm{d}V_d|$). The expectation on the righthand side of equation (5.2) is the conditional expectation of

$$|\det^{\perp}(\nabla_x^d s_d)|\,|\det^{\perp}(\nabla_y^d s_d)|,$$

given that $\mathrm{ev}_{x,y}^d(s_d) = 0$.

Proof. This formula was proved in [22, Theorem 4.4] in the case $1 \leqslant r < n$. The hypothesis $r < n$ does not appear in the proof and can be changed to $r \leqslant n$ without any other modification. □

Definition 5.6. Let $d \geqslant \max(d_1, d_2)$, and let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$. We denote by $\mathcal{D}_d : M^2 \setminus \Delta \to \mathbb{R}$ the map defined by

$$\mathcal{D}_d(x,y) = \left(\frac{\mathbb{E}[|\det^{\perp}(\nabla^d_x s_d)|\, |\det^{\perp}(\nabla^d_y s_d)| : s_d(x) = 0 = s_d(y)]}{|\det^{\perp}(\mathrm{ev}^d_{x,y})|} \right. \\ \left. - \frac{\mathbb{E}[|\det^{\perp}(\nabla^d_x s_d)| : s_d(x) = 0]\, \mathbb{E}[|\det^{\perp}(\nabla^d_y s_d)| : s_d(y) = 0]}{|\det^{\perp}(\mathrm{ev}^d_x)|\, |\det^{\perp}(\mathrm{ev}^d_y)|} \right).$$

Remark 5.7. Note that $\mathcal{D}_d$ does not depend on the choice of ∇^d. Indeed, we only consider derivatives of s_d at points where it vanishes.

Proposition 5.8. *For all $d \geqslant \max(d_1, d_2)$, we have for any $\phi_1, \phi_2 \in C^0(M)$*

$$\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = \frac{1}{(2\pi)^r} \int_{M^2 \setminus \Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 \\ + \delta_{rn}\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1\phi_2 \rangle],$$

where δ_{rn} equals 1 if $r = n$ and 0 otherwise.

Proof. This was proved in [22, Section 4.2] for $r < n$ (the case $r = n$ requires an extra argument). The following proof is valid for any $r \in \{1, \dots, n\}$. Let ϕ_1 and $\phi_2 \in C^0(M)$; we have

$$\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = \mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1 \rangle \langle |\mathrm{d}V_d|, \phi_2 \rangle] \\ - \mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1 \rangle]\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_2 \rangle]. \tag{5.3}$$

Since Z_d has almost surely dimension $n - r$, the diagonal in $Z_d \times Z_d$ is negligible if and only if $r < n$. Moreover, if $r = n$ then both $|\mathrm{d}V_d|$ and $|\mathrm{d}V_d|^2$ are counting measures. Then,

$$\langle |\mathrm{d}V_d|, \phi_1 \rangle \langle |\mathrm{d}V_d|, \phi_2 \rangle = \int_{(x,y)\in(Z_d)^2} \phi_1(x)\phi_2(y)\, |\mathrm{d}V_d|^2 \\ = \int_{(x,y)\in(Z_d)^2\setminus\Delta} \phi_1(x)\phi_2(y)\, |\mathrm{d}V_d|^2 + \delta_{rn} \int_{x \in Z_d} \phi_1(x)\phi_2(x)\, |\mathrm{d}V_d|,$$

almost surely. Hence,

$$\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1 \rangle \langle |\mathrm{d}V_d|, \phi_2 \rangle] \\ = \mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta} \phi_1(x)\phi_2(y)\, |\mathrm{d}V_d|^2 \right] + \delta_{rn}\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1\phi_2 \rangle]. \tag{5.4}$$

We apply Theorem 5.5 to the first term on the righthand side of equation (5.4). Similarly, we apply Theorem 5.1 to $\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_i\rangle]$ for $i \in \{1,2\}$. This yields the result by equation (5.3). ◻

By Theorem 1.2, if $r = n$, for all $\phi_1, \phi_2 \in C^0(M)$ we have

$$\begin{aligned}&\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1\phi_2\rangle]\\ &\qquad = d^{n/2}\frac{2}{\mathrm{Vol}(\mathbb{S}^n)}\left(\int_M \phi_1\phi_2 \,|\mathrm{d}V_M|\right) + \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{n/2-1}).\end{aligned}$$

Hence, in order to prove Theorem 1.6, we have to show that, for any $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$,

$$\begin{aligned}(5.5)\quad &\int_{M^2\setminus\Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\\ &\quad = d^{r-n/2}\left(\int_M \phi_1\phi_2\,|\mathrm{d}V_M|\right)\mathrm{Vol}(\mathbb{S}^{n-1})\mathcal{I}_{n,r}\\ &\qquad + \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{r-n/2-\alpha}) + \|\phi_1\|_\infty \varpi_{\phi_2}(C_\beta d^{-\beta})O(d^{r-n/2}),\end{aligned}$$

where α, β, C_β, and $\mathcal{I}_{n,r}$ are as in Theorem 1.6.

This is done in two steps. The mass of the integral on the lefthand side of equation (5.5) concentrates in a neighborhood of Δ of typical size $1/\sqrt{d}$. More specifically, let us now fix the value of the constant b_n appearing in Proposition 3.4.

Definitions 5.9. We set $b_n = (1/C_2)(n/2+1)$, where $C_2 > 0$ is the constant appearing in the exponential in Theorem 3.6. Moreover, for all $d \in \mathbb{N}^*$, we denote

$$\Delta_d = \left\{(x,y) \in M^2 \mid \rho_g(x,y) < b_n\frac{\ln d}{\sqrt{d}}\right\},$$

where, ρ_g is the geodesic distance in (M,g).

In Section 5.3 below, we show that, in equation (5.5), the integral over $M^2\setminus\Delta_d$ only contributes what turns out to be an error term. We refer to this term as the *far off-diagonal term*. In Subsection 5.4 we complete the proof of (5.5) by studying the *near-diagonal term*, that is, the integral of $\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)$ over $\Delta_d \setminus \Delta$. This turns out to be the leading term.

5.2. Expression of some covariances. To prove (5.5), we need to study the distribution of the random variables appearing in the definition of $\mathcal{D}_d$ (see Definition 5.6). The purpose of this section is to introduce several variance operators that will appear in the proof. In the following, ∇^d denotes a real connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$.

5.2.1. Uncorrelated terms. First of all, let us consider the distribution of $s_d(x)$ for any $x \in M$. Since $s_d \sim \mathcal{N}(\mathrm{Id})$ and ev_x^d is linear (see Definition 2.2), $s_d(x)$ is a centered Gaussian vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ with variance operator $\mathrm{ev}_x^d(\mathrm{ev}_x^d)^* = E_d(x,x)$.

Lemma 5.10. *For all $x \in M$, we have*

$$|\det{}^{\perp}(\mathrm{ev}_x^d)| = \left(\frac{d}{\pi}\right)^{rn/2}(1 + O(d^{-1})),$$

where the error term is independent of x.

Proof. We have $|\det^{\perp}(\mathrm{ev}_x^d)| = \det(E_d(x,x))^{1/2}$, and by Theorem 3.3,

$$E_d(x,x) = \left(\frac{d}{n}\right)^{n}(\mathrm{Id} + O(d^{-1})). \qquad \square$$

Corollary 5.11. *There exists $d_1 \in \mathbb{N}$ such that, for all $d \geqslant d_1$, for all $x \in M$, ev_x^d is surjective: that is, $(s_d(x))$ is non-degenerate.*

Then, let $d \in \mathbb{N}$ and $x \in M$. We denote by $j_x^d : s \mapsto (s(x), \nabla_x^d s)$ the evaluation of the 1-jet of a section at the point x. The distribution of the random vector $(s_d(x), \nabla_x^d s_d)$ is a centered Gaussian in

$$\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus (\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes T_x^*M),$$

with variance operator

$$\begin{aligned} j_x^d(j_x^d)^* &= \mathbb{E}[j_x^d(s_d) \otimes j_x^d(s_d)^*] \\ &= \begin{pmatrix} \mathbb{E}[s_d(x) \otimes s_d(x)^*] & \mathbb{E}[s_d(x) \otimes (\nabla_x^d s_d)^*] \\ \mathbb{E}[(\nabla_x^d s_d) \otimes s_d(x)^*] & \mathbb{E}[(\nabla_x^d s_d) \otimes (\nabla_x^d s_d)^*] \end{pmatrix} \\ &= \begin{pmatrix} E_d(x,x) & \partial_y^\sharp E_d(x,x) \\ \partial_x E_d(x,x) & \partial_x \partial_y^\sharp E_d(x,x) \end{pmatrix}. \end{aligned}$$

If $d \geqslant d_1$, then $s_d(x)$ is non-degenerate, and the distribution of $\nabla_x^d s_d$, given that $s_d(x) = 0$, is a centered Gaussian whose variance equals

$$\partial_x \partial_y^\sharp E_d(x,x) - \partial_x E_d(x,x)(E_d(x,x))^{-1}\partial_y^\sharp E_d(x,x).$$

By Theorem 3.3, this variance equals

$$\frac{d^{n+1}}{\pi^n}(\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_x^*M} + O(d^{-1}))$$

as d goes to infinity and the error does not depend on x.

Remark 5.12. If (s, x) is such that $s(x) = 0$, then $\nabla^d_x s$ does not depend on the connection ∇^d. This explains why the distribution of $\nabla^d_x s_d$ given that $s_d(x) = 0$, in particular its variance, does not depend on ∇^d.

Lemma 5.13. *For every $x \in M$, we have*

$$\begin{aligned}\mathbb{E}[|\det{}^{\perp}(\nabla^d_x s_d)| : s_d(x) = 0] \\ = \left(\frac{d^{n+1}}{\pi^n}\right)^{r/2} (2\pi)^{r/2} \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)} (1 + O(d^{-1})),\end{aligned}$$

where the error term is independent of x.

Proof. This was proved in [22, Lemma 4.7] for $r < n$. The proof is the same here. ∎

5.2.2. Correlated terms far from the diagonal. Let us now focus on variables where non-trivial correlations may appear in the limit. Let $d \in \mathbb{N}$, for all $(x, y) \in M^2 \setminus \Delta$; the random vector $\mathrm{ev}^d_{x,y}(s_d) = (s_d(x), s_d(y))$ is a centered Gaussian vector with variance operator

$$\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^* = \mathbb{E}[\mathrm{ev}^d_{x,y}(s_d) \otimes \mathrm{ev}^d_{x,y}(s_d)^*] = \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix}, \tag{5.6}$$

where we decomposed this operator according to the direct sum

$$\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_y.$$

Definition 5.14. For all $d \in \mathbb{N}$, for all $(x, y) \in M^2 \setminus \Delta$, we denote by

$$\Theta_d(x, y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix}$$

the variance of the centered Gaussian vector $(\pi/d)^{n/2}(s_d(x), s_d(y))$.

Note that, by Lemma 5.4, for all $d \geqslant d_2$, $\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*$ is non-singular: that is, $(s_d(x), s_d(y))$ is non-degenerate and $\Theta_d(x, y)$ is non-singular.

Let $d \in \mathbb{N}$ and $(x, y) \in M^2 \setminus \Delta$. We denote the evaluation of the 1-jets at (x, y) by $j^d_{x,y} : s \mapsto (s(x), s(y), \nabla^d_x s, \nabla^d_y s)$. Then, we have that

$$j^d_{x,y}(s_d) = (s_d(x), s_d(y), \nabla^d_x s_d, \nabla^d_y s_d)$$

is a centered Gaussian vector in

$$\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_y \oplus (\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes T^*_x M) \oplus (\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_y \otimes T^*_y M),$$

whose variance operator $j^d_{x,y}(j^d_{x,y})^*$ equals

$$\begin{aligned}
&\mathbb{E}[j^d_{x,y}(s_d)\otimes(j^d_{x,y}(s_d))^*]\\
&\quad=\begin{pmatrix}
E_d(x,x) & E_d(x,y) & \partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y)\\
E_d(y,x) & E_d(y,y) & \partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y)\\
\partial_x E_d(x,x) & \partial_x E_d(x,y) & \partial_x\,\partial^\sharp_y E_d(x,x) & \partial_x\,\partial^\sharp_y E_d(x,y)\\
\partial_x E_d(y,x) & \partial_x E_d(y,y) & \partial_x\,\partial^\sharp_y E_d(y,x) & \partial_x\,\partial^\sharp_y E_d(y,y)
\end{pmatrix}.
\end{aligned}$$

Definition 5.15. For all $d\in\mathbb{N}$, for all $(x,y)\in M^2\setminus\Delta$, we denote by

$$\begin{aligned}
\Omega_d(x,y)&=\left(\frac{\pi}{d}\right)^n\\
&\times\begin{pmatrix}
E_d(x,x) & E_d(x,y) & d^{-1/2}\,\partial^\sharp_y E_d(x,x) & d^{-1/2}\,\partial^\sharp_y E_d(x,y)\\
E_d(y,x) & E_d(y,y) & d^{-1/2}\,\partial^\sharp_y E_d(y,x) & d^{-1/2}\,\partial^\sharp_y E_d(y,y)\\
d^{-1/2}\,\partial_x E_d(x,x) & d^{-1/2}\,\partial_x E_d(x,y) & d^{-1}\,\partial_x\,\partial^\sharp_y E_d(x,x) & d^{-1}\,\partial_x\,\partial^\sharp_y E_d(x,y)\\
d^{-1/2}\,\partial_x E_d(y,x) & d^{-1/2}\,\partial_x E_d(y,y) & d^{-1}\,\partial_x\,\partial^\sharp_y E_d(y,x) & d^{-1}\,\partial_x\,\partial^\sharp_y E_d(y,y)
\end{pmatrix}
\end{aligned}$$

the variance operator of the centered Gaussian vector

$$\left(\frac{\pi}{d}\right)^{n/2}\left(s_d(x),s_d(y),\frac{1}{\sqrt{d}}\nabla^d_x s_d,\frac{1}{\sqrt{d}}\nabla^d_y s_d\right).$$

Let us now assume that $d\geqslant d_2$, so that the distribution of $(s_d(x),s_d(y))$ is non-degenerate. Then, the distribution of $(\nabla^d_x s,\nabla^d_y s)$, given that $s_d(x)=0=s_d(y)$, is a centered Gaussian with variance operator

$$\begin{aligned}
&\begin{pmatrix}
\partial_x\,\partial^\sharp_y E_d(x,x) & \partial_x\,\partial^\sharp_y E_d(x,y)\\
\partial_x\,\partial^\sharp_y E_d(y,x) & \partial_x\,\partial^\sharp_y E_d(y,y)
\end{pmatrix}-\begin{pmatrix}
\partial_x E_d(x,x) & \partial_x E_d(x,y)\\
\partial_x E_d(y,x) & \partial_x E_d(y,y)
\end{pmatrix}\\
&\qquad\times\begin{pmatrix}
E_d(x,x) & E_d(x,y)\\
E_d(y,x) & E_d(y,y)
\end{pmatrix}^{-1}\begin{pmatrix}
\partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y)\\
\partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y)
\end{pmatrix}.
\end{aligned}$$

Definition 5.16. For all $d\geqslant d_2$, for all $(x,y)\in M^2\setminus\Delta$, we set

$$\begin{aligned}
\Lambda_d(x,y)=\frac{\pi^n}{d^{n+1}}&\left(\begin{pmatrix}
\partial_x\,\partial^\sharp_y E_d(x,x) & \partial_x\,\partial^\sharp_y E_d(x,y)\\
\partial_x\,\partial^\sharp_y E_d(y,x) & \partial_x\,\partial^\sharp_y E_d(y,y)
\end{pmatrix}\right.\\
&-\begin{pmatrix}
\partial_x E_d(x,x) & \partial_x E_d(x,y)\\
\partial_x E_d(y,x) & \partial_x E_d(y,y)
\end{pmatrix}\begin{pmatrix}
E_d(x,x) & E_d(x,y)\\
E_d(y,x) & E_d(y,y)
\end{pmatrix}^{-1}\\
&\left.\times\begin{pmatrix}
\partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y)\\
\partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y)
\end{pmatrix}\right),
\end{aligned}$$

which is the variance of the Gaussian vector $(\pi^n/d^{n+1})^{1/2}(\nabla^d_x s_d, \nabla^d_y s_d)$, given that $s_d(x) = 0 = s_d(y)$.

Remark 5.17. Once again, $\Lambda_d(x, y)$ is independent of the choice of ∇^d, and so is the distribution of $(\nabla^d_x s_d, \nabla^d_y s_d)$ given that $s_d(x) = 0 = s_d(y)$. On the other hand, the distribution of $(s_d(x), s_d(y), \nabla^d_x s_d, \nabla^d_y s_d)$ heavily depends on ∇^d, and so does $\Omega_d(x, y)$. Hence, we will need to specify a choice of ∇^d at some point when dealing with Ω_d.

5.2.3. Correlated terms close to the diagonal. Finally, we need to consider the distribution of the 1-jets of s_d at x and $y \in M$, when the distance between x and y is of order $1/\sqrt{d}$. As in Section 3, let $R > 0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. There exists $d_3 \in \mathbb{N}$ such that, for all $d \geqslant d_3$, we have $b_n \ln d/\sqrt{d} \leqslant R$.

Let $d \geqslant d_3$ and let $(x, y) \in \Delta_d \setminus \Delta$. Using the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ around x (see Subsection 3.1), we can see $(\pi/d)^{n/2}(s_d(x), s_d(y))$ as a random vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$. Since the distance from x to y is smaller than the injectivity radius of M, we can write y as $\exp_x(z/\sqrt{d})$ for some $z \in T_xM$. Moreover, $\|z\| = \sqrt{d}\rho_g(x, y) < b_n \ln d$.

Definition 5.18. Let $d \geqslant d_3$, let $x \in M$, and let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We set

$$\Theta_d(z) = \Theta_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ via the real normal trivialization centered at x.

Remark 5.19. Beware that $\Theta_d(z)$ depends on x, even if this is not reflected in the notation. However, we will show that the limit of $\Theta_d(z)$ as $d \to +\infty$ does not depend on x.

Recall that e_d was defined by equation (3.1) as a map from $T_xM \times T_xM$ to $\mathrm{End}(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)$. The definitions of $\Theta_d(x, y)$ (Definition 5.14) and e_d show that, for all $d \geqslant d_3$, for all $x \in M$, and for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$,

$$\Theta_d(z) = \begin{pmatrix} e_d(0,0) & e_d(0,z) \\ e_d(z,0) & e_d(z,z) \end{pmatrix}. \tag{5.7}$$

We can define $\Omega_d(z)$ and $\Lambda_d(z)$ similarly and express them in terms of e_d and its derivatives.

Definition 5.20. Let $d \geqslant d_3$, let $x \in M$, and let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We set

$$\Omega_d(z) = \Omega_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $(\mathbb{R} \oplus \mathbb{R} \oplus T^*_xM \oplus T^*_xM) \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ via the real normal trivialization centered at x.

Let ∇^d be a real connection on $\mathcal{E} \otimes \mathcal{L}^d$ such that, in the real normal trivialization around x, this connection coincides over the ball $B_{T_x\mathcal{X}}(0,R)$ with the usual differentiation for maps from $T_x\mathcal{X}$ to $(\mathcal{E} \otimes \mathcal{L}^d)_x$. The existence of such a connection was established at the end of Subsection 3.1. Then, by Definitions 5.15 and 5.20, we have for all $d \geqslant d_3$, for all $x \in M$, and for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ that

$$\Omega_d(z) = \begin{pmatrix} e_d(0,0) & e_d(0,z) & \partial_y^\sharp e_d(0,0) & \partial_y^\sharp e_d(0,z) \\ e_d(z,0) & e_d(z,z) & \partial_y^\sharp e_d(z,0) & \partial_y^\sharp e_d(z,z) \\ \partial_x e_d(0,0) & \partial_x e_d(0,z) & \partial_x \partial_y^\sharp e_d(0,0) & \partial_x \partial_y^\sharp e_d(0,z) \\ \partial_x e_d(z,0) & \partial_x e_d(z,z) & \partial_x \partial_y^\sharp e_d(z,0) & \partial_x \partial_y^\sharp e_d(z,z) \end{pmatrix}. \tag{5.8}$$

Definition 5.21. Let $d \geqslant \max(d_2, d_3)$, $x \in M$, $z \in B_{T_x^*M}(0, b_n \ln d) \setminus \{0\}$. We set

$$\Lambda_d(z) = \Lambda_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $(T_x^*M \oplus T_x^*M) \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ via the real normal trivialization around x.

Then, we have

$$\Lambda_d(z) = \begin{pmatrix} \partial_x \partial_y^\sharp e_d(0,0) & \partial_x \partial_y^\sharp e_d(0,z) \\ \partial_x \partial_y^\sharp e_d(z,0) & \partial_x \partial_y^\sharp e_d(z,z) \end{pmatrix} - \\ - \begin{pmatrix} \partial_x e_d(0,0) & \partial_x e_d(0,z) \\ \partial_x e_d(z,0) & \partial_x e_d(z,z) \end{pmatrix} \Theta_d(z)^{-1} \begin{pmatrix} \partial_y^\sharp e_d(0,0) & \partial_y^\sharp e_d(0,z) \\ \partial_y^\sharp e_d(z,0) & \partial_y^\sharp e_d(z,z) \end{pmatrix},$$

for all $d \geqslant \max(d_2, d_3)$, all $x \in M$, and all $z \in B_{T_x^*M}(0, b_n \ln d) \setminus \{0\}$.

5.3. Far off-diagonal term. In this section, we state that the far off-diagonal term in equation (5.5) only contributes an error term. This was already proved in [22] for $r < n$. The proof is the same for $r = n$, so we refer to [22] for the proof. Lemma 5.23 below is used in the proof of Proposition 5.22, but is also of independent interest for our purpose.

Proposition 5.22. *Let $\phi_1, \phi_2 \in C^0(M)$. Then, as $d \to +\infty$ we have the following:*

$$\int_{M^2 \setminus \Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y) \, |\mathrm{d}V_M|^2 = \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{r-n/2-1}),$$

where the error term is independent of (ϕ_1, ϕ_2).

Lemma 5.23. *For every* $(x, y) \in M^2 \setminus \Delta_d$, *we have.*

$$\Theta_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} (\mathrm{Id} + O(d^{-(n/2)-1})),$$

where the error term is independent of $(x, y) \in M^2 \setminus \Delta_d$.

Proof. Since $(x, y) \in M^2 \setminus \Delta_d$, we have $\rho_g(x, y) \geqslant b_n \ln d/\sqrt{d}$. With our choice of b_n (see Definition 5.9), the error term in Theorem 3.6 is then $O(d^{(n-k)/2-1})$, uniformly on $M^2 \setminus \Delta_d$. Thus, by Theorem 3.6,

$$\Theta_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} + O(d^{-n/2-1}).$$

The result follows from the fact that the leading term is $\mathrm{Id} + O(d^{-1})$, by Theorem 3.3. □

5.4. Near-diagonal term. In this section, we conclude the proof of Theorem 1.6, up to the technical lemmas whose proofs were postponed until Appendices A and B.

Definition 5.24. Let

$$d \geqslant \max(d_1, d_2, d_3), \quad x \in M, \quad z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}.$$

We define

$$\mathrm{D}_d(x,z) = d^{-r}\mathcal{D}_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right).$$

Recall that $\mathrm{D}_{n,r}$ was defined by Definition 4.15. The main result of this section is the following.

Proposition 5.25. *Let* $\alpha \in (0, 1)$.
Then, for all $x \in M$, *for all* $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ *we have*

$$\mathrm{D}_d(x,z) = \mathrm{D}_{n,r}(\|z\|^2)(1 + O(d^{-\alpha})) + O(d^{-\alpha}),$$

where the error terms do not depend on (x, z).

First, let us prove that Propositions 5.8, 5.22, and 5.25 together imply Theorem 1.6.

Proof of Theorem 1.6. The main point is to compute the asymptotic of the near-diagonal term in equation (5.5). Let us fix $\alpha \in (0, 1)$, $\beta \in (0, \frac{1}{2})$, and $\phi_1, \phi_2 \in C^0(M)$. Let $x \in M$, and recall that $\sqrt{\kappa}$ is the density of $|\mathrm{d}V_M|$ with respect to the Lebesgue measure, in the exponential chart centered at x, where κ was introduced in Subsection 3.2. Then, by a change of variable $y = \exp_x(z/\sqrt{d})$, we have

$$\begin{aligned}(5.9)\quad &\int_{\Delta_d\setminus\Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\\ &\quad= d^{r-n/2}\int_{x\in M}\phi_1(x)\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)\mathrm{D}_d(x,z)\\ &\qquad\times\kappa\left(\frac{z}{\sqrt{d}}\right)^{1/2}\mathrm{d}z\,|\mathrm{d}V_M|.\end{aligned}$$

As we already discussed in Subsection 3.2, $\kappa(z) = 1 + O(\|z\|^2)$, and the error term is independent of x. Hence, $\kappa(z/\sqrt{d})^{1/2} = 1 + O((\ln d)^2/d)$, and by Proposition 5.25, for all $\gamma \in (\alpha, 1)$,

$$\begin{aligned}(5.10)\quad &\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)\mathrm{D}_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{1/2}\mathrm{d}z\\ &\quad=\left(\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)\mathrm{D}_{n,r}(\|z\|^2)\,\mathrm{d}z\right)(1+O(d^{-\gamma}))\\ &\qquad+\|\phi_2\|_\infty O\left(\frac{(\ln d)^n}{d^\gamma}\right).\end{aligned}$$

Since $\gamma > \alpha$, $(\ln d)^n d^{-\gamma} = O(d^{-\alpha})$. Similarly, there exists $C_\beta > 0$ such that $b_n \ln d/\sqrt{d} \leqslant C_\beta d^{-\beta}$ for all $d \in \mathbb{N}^*$. Then, we have

$$\begin{aligned}(5.11)\quad &\left|\int_{z\in B_{T_xM}(0,b_n\ln d)}\left(\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)-\phi_2(x)\right)\mathrm{D}_{n,r}(\|z\|^2)\,\mathrm{d}z\right|\\ &\quad\leqslant\varpi_{\phi_2}(C_\beta d^{-\beta})\int_{z\in B_{T_xM}(0,b_n\ln d)}|\mathrm{D}_{n,r}(\|z\|^2)|\,\mathrm{d}z,\end{aligned}$$

where ϖ_{ϕ_2} is the continuity modulus of ϕ_2 (see Definition 1.3). Besides, by Corollary 4.19,

$$\begin{aligned}(5.12)\quad &\int_{z\in B_{T_xM}(0,b_n\ln d)}|\mathrm{D}_{n,r}(\|z\|^2)|\,\mathrm{d}z\\ &\quad=\mathrm{Vol}(\mathbb{S}^{n-1})\frac{1}{2}\int_{t=0}^{(b_n\ln d)^2}\mathrm{D}_{n,r}(t)t^{(n-2)/2}\,\mathrm{d}t\\ &\quad=\mathrm{Vol}(\mathbb{S}^{n-1})(\mathcal{I}_{n,r}+O(e^{-(1/4)(b_n\ln d)^2})),\end{aligned}$$

and the error term is $O(d^{-1})$, since $(1/4)(b_n \ln d)^2 \geqslant \ln d$ for d large enough. By equations (5.10), (5.11), and (5.12), the innermost integral on the righthand side of equation (5.9) equals

$$\phi_2(x)\,\mathrm{Vol}(\mathbb{S}^{n-1})\mathcal{I}_{n,r} + \varpi_{\phi_2}(C_\beta d^{-\beta})O(1) + \|\phi_2\|_\infty O(d^{-\alpha}),$$

and the error terms are independent of $x \in M$ and (ϕ_1, ϕ_2). Finally, by equation (5.9),

$$\int_{\Delta_d \setminus \Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2 \\ = d^{r-n/2}\left(\int_M \phi_1\phi_2\,|\mathrm{d}V_M|\right)\mathrm{Vol}(\mathbb{S}^{n-1})\mathcal{I}_{n,r} \\ + \|\phi_1\|_\infty \varpi_{\phi_2}(C_\beta d^{-\beta})O(d^{r-n/2}) + \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{r-n/2-\alpha}).$$

We conclude the proof by combining this last relation with Proposition 5.8, 5.22, and, in the case $r = n$, Theorem 1.2 for $\phi_1\phi_2$. ◻

The remainder of this section is mostly dedicated to the proof of Proposition 5.25. We will deduce this proposition from several technical lemmas stated below.

Let $x \in M$; then, any choice of an isometry between T_xM and $\mathbb{R}^n$ and an isometry between $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ and $\mathbb{R}^r$ allows us to see the Bargmann-Fock process $(s(z))_{z\in\mathbb{R}^n}$, studied in Section 4, as a smooth Gaussian process from T_xM to $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. The distribution of this process does not depend on our choice of isometries. Thus, in the following, we can consider $\Theta(z)$ and $\Theta_d(z)$ (respectively, $\Omega(z)$ and $\Omega_d(z)$, respectively $\Lambda(z)$ and $\Lambda_d(z)$) as operators on the same space.

Lemma 5.26. *Let $\alpha \in (0,1)$; then, we have*

$$\Theta(z)^{-1/2}\Theta_d(z)\Theta(z)^{-1/2} = \mathrm{Id} + O(d^{-\alpha}),$$

for all $x \in M$ and all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. The error term does not depend on (x,z).

Proof. See Appendix B. ◻

Remark 5.27. It might be wondered why Lemma 5.26 does not state that $\Theta_d(z) = \Theta(z)(\mathrm{Id} + O(d^{-\alpha}))$, which would be somewhat simpler. First, note this statement is not equivalent to Lemma 5.26, since some of the eigenvalues of $\Theta(z)$ converge to 0 as $z \to 0$. In fact, this alternative statement turns out to be false in general. Moreover, even if $\Theta_d(z)$ is a linear map, it represents a variance, that is, something intrinsically bilinear. It is then quite natural to consider

$$\Theta(z)^{-1/2}\Theta_d(z)\Theta(z)^{-1/2}$$

since this is how $\Theta_d(z)$ transforms if we act on $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ by $\Theta(z)^{1/2}$. This remark also applies to Lemmas 5.28 and 5.29 below.

Let us forget about the proof of Proposition 5.25 for a minute, and prove the existence of d_2 (see Lemma 5.4) as a corollary of Lemmas 5.23 and 5.26. Note that the proofs of these lemmas only rely on the estimates of Section 3, so there is no logical loop here.

Proof of Lemma 5.4. We want to prove that, as soon as d is large enough, $\mathrm{ev}^d_{x,y}$ is surjective for all $(x,y) \in M^2 \setminus \Delta$; that is, $\det(\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*) \neq 0$. By equation (5.6) and the definition of Θ_d (Definition 5.14),

$$\det(\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*) = \left(\frac{d}{\pi}\right)^{2rn} \det\left(\Theta_d(x,y)\right),$$

so we have to show that $\det(\Theta_d(x,y))$ does not vanish on $M^2 \setminus \Delta$, for d large enough. By Lemma 5.23 and Theorem 3.3,

$$\det(\Theta_d(x,y)) = 1 + O(d^{-1}), \tag{5.13}$$

uniformly on $M^2 \setminus \Delta_d$. Let $(x,y) \in \Delta_d \setminus \Delta$ and let us assume that $d \geqslant d_3$ so that we can write y as $\exp_x(z/\sqrt{d})$ with $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Then, by Lemmas 5.26 and 4.8,

$$\begin{aligned} \det(\Theta_d(x,y)) = \det(\Theta_d(z)) &= \det(\Theta(z))(1 + O(\sqrt{d})) \\ &= (1 - e^{-\|z\|^2})^r(1 + O(\sqrt{d})), \end{aligned} \tag{5.14}$$

uniformly on $\Delta_d \setminus \Delta$. The result follows from equations (5.13) and (5.14). ◻

We can now go back to the proof of Proposition 5.25.

Lemma 5.28. *Let $\alpha \in (0,1)$. Then, we have*

$$\Omega(z)^{-1/2}\Omega_d(z)\Omega(z)^{-1/2} = \mathrm{Id} + O(d^{-\alpha}),$$

for all $x \in M$ and all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. The error term does not depend on (x,z).

Proof. See Appendix B. ◻

Lemma 5.29. *Let $\alpha \in (0,1)$; then, we have*

$$\Lambda(z)^{-1/2}\Lambda_d(z)\Lambda(z)^{-1/2} = \mathrm{Id} + O(d^{-\alpha}),$$

for all $x \in M$ and all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. The error term does not depend on (x,z).

Proof. Let $\alpha \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. By Definition 5.20, $\Omega_d(z)$ is an operator on

$$(\mathbb{R}^2 \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x) \oplus ((T^*_xM)^2 \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x).$$

Using this splitting, we write $\Omega_d(z)$ by blocks as

$$\Omega_d(z) = \begin{pmatrix} \Theta_d(z) & \Omega^1_d(z)^* \\ \Omega^1_d(z) & \Omega^2_d(z) \end{pmatrix},$$

thus defining $\Omega_d^1(z)$ and $\Omega_d^2(z)$. For d large enough, $\Theta_d(z)$ is invertible and its Schur complement is $\Lambda_d(z) = \Omega_d^2(z) - \Omega_d^1(z)\Theta_d(z)^{-1}\Omega_d^1(z)^*$. It is then known that $\Lambda_d(z)^{-1}$ is the bottom-right block of $\Omega_d(z)^{-1}$, that is,

$$\begin{aligned}\Lambda_d(z)^{-1} &= \begin{pmatrix}0 & \mathrm{Id}\end{pmatrix} \Omega_d(z)^{-1} \begin{pmatrix}0 \\ \mathrm{Id}\end{pmatrix} \\ &= \begin{pmatrix}0 & \mathrm{Id}\end{pmatrix} (\Omega(z)^{-1} + \Omega(z)^{-1/2}O(d^{-\alpha})\Omega(z)^{-1/2}) \begin{pmatrix}0 \\ \mathrm{Id}\end{pmatrix},\end{aligned}$$

where the second equality is given by Lemma 5.28 and the error term is independent of (x, z). Similarly, $\Lambda(z)$ is the Schur complement of $\Theta(z)$ in $\Omega(z)$, so that

$$\Lambda(z)^{-1} = \begin{pmatrix}0 & \mathrm{Id}\end{pmatrix} \Omega(z)^{-1} \begin{pmatrix}0 \\ \mathrm{Id}\end{pmatrix}.$$

Moreover, by Lemma 4.14,

$$\begin{pmatrix}0 & \Lambda(z)^{1/2}\end{pmatrix} \Omega_d(z)^{-1/2} \quad \text{is bounded.}$$

Hence, $\Lambda(z)^{1/2}\Lambda_d(z)^{-1}\Lambda(z)^{1/2} = \mathrm{Id} + O(d^{-\alpha})$, and the error term still does not depend on (x, z). □

Lemma 5.30. *Let $\alpha \in (0, 1)$, let $x \in M$, and let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Let $X_d(z)$ and $Y_d(z)$ be random vectors in $T_x^*M \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ such that*

$$(X_d(z), Y_d(z)) \sim \mathcal{N}(\Lambda_d(z)).$$

Then, we have

$$\begin{aligned}&\mathbb{E}[|\det{}^\perp(X_d(z))|\,|\det{}^\perp(Y_d(z))|] \\ &\qquad = \mathbb{E}[|\det{}^\perp(X_\infty(z))|\,|\det{}^\perp(Y_\infty(z))|](1 + O(d^{-\alpha})),\end{aligned}$$

where $(X_\infty(z), Y_\infty(z)) \sim \mathcal{N}(\Lambda(z))$ and the error term does not depend on (x, z).

Proof. See Appendix B. □

We conclude this section with the proof of Proposition 5.25. Recall the definitions of $\mathrm{D}_{n,r}(t)$ (Definition 4.15), $\mathcal{D}_d(x, y)$ (Definition 5.6), and $\mathrm{D}_d(x, z)$ (Definition 5.24).

Proof of Proposition 5.25. Let us fix $\alpha \in (0, 1)$.

Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Then, we set $y = \exp_x(z/\sqrt{d})$. We have defined $\Theta_d(z)$ and $\Lambda_d(z)$ so that

$$\frac{1}{d^r} \frac{\mathbb{E}[|\det^{\perp}(\nabla^d_x s_d)|\,|\det^{\perp}(\nabla^d_y s_d)| : s_d(x) = 0 = s_d(y)]}{|\det^{\perp}(\mathrm{ev}^d_{x,y})|} = \frac{\mathbb{E}[|\det^{\perp}(X_d(z))|\,|\det^{\perp}(Y_d(z))|]}{\det(\Theta_d(z))^{1/2}},$$

where $(X_d(z), Y_d(z)) \sim \mathcal{N}(\Lambda_d(z))$. By Lemmas 5.26 and 5.30, this equals

$$\frac{\mathbb{E}[|\det^{\perp}(X_\infty(z))|\,|\det^{\perp}(Y_\infty(z))|]}{\det(\Theta(z))^{1/2}}(1 + O(d^{-\alpha})),$$

where $(X_\infty(z), Y_\infty(z)) \sim \mathcal{N}(\Lambda(z))$ and the error term does not depend on (x, z). $(X_\infty(z), Y_\infty(z))$ is distributed as $(d_0 s, d_z s)$, where s is a copy of the Bargmann-Fock process from T_xM to $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$, by the definition of $\Lambda(z)$ (cf. Subsection 4.3). Then, by Lemma 4.16,

$$\mathbb{E}[|\det^{\perp}(X_\infty(z))|\,|\det^{\perp}(Y_\infty(z))|] = \mathbb{E}[|\det^{\perp}(X(\|z\|^2))|\,|\det^{\perp}(Y(\|z\|^2))|],$$

where $(X(\|z\|^2), Y(\|z\|^2))$ was defined by Definition 1.5. Besides, $\det(\Theta(z)) = (1 - e^{-\|z\|^2})^r$ by Lemma 4.1, so that

$$\frac{1}{d^r} \frac{\mathbb{E}[|\det^{\perp}(\nabla^d_x s_d)|\,|\det^{\perp}(\nabla^d_y s_d)| : s_d(x) = 0 = s_d(y)]}{|\det^{\perp}(\mathrm{ev}^d_{x,y})|} = \left(\mathrm{D}_{n,r}(\|z\|^2) + (2\pi)^r \left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2\right)(1 + O(d^{-\alpha})).$$

Besides, by Lemmas 5.10 and 5.13,

$$\frac{1}{d^r} \frac{\mathbb{E}[|\det^{\perp}(\nabla^d_x s_d)| : s_d(x) = 0]}{|\det^{\perp}(\mathrm{ev}^d_x)|} \frac{\mathbb{E}[|\det^{\perp}(\nabla^d_y s_d)| : s_d(y) = 0]}{|\det^{\perp}(\mathrm{ev}^d_x)|} = (2\pi)^r \left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2 + O(d^{-1}).$$

This yields the desired relation. □

6. PROOF OF THEOREM 1.8

The goal of this section is to prove that the leading constant in Theorem 1.6 is positive. Subsection 6.1 is concerned with the definition and proprieties of Kostlan-Shub-Smale polynomials. In Subsection 6.2 we recall some facts about Wiener chaoses, while in Subsection 6.3 we compute the chaotic expansion of the linear statistics $\langle |\mathrm{d}V_d|, \phi\rangle$. Finally, we conclude the proof of Theorem 1.8 in Subsection 6.4.

6.1. Kostlan-Shub-Smale polynomials. Here, we describe a special case of our real algebraic framework. This is the setting we will be considering throughout the proof of Theorem 1.8. A good reference for the complex algebraic material of this section is [16].

6.1.1. Definition. We choose $\mathcal{X}$ to be the complex projective space $\mathbb{CP}^n$ with the real structure induced by the complex conjugation in $\mathbb{C}^{n+1}$. The real locus of $\mathcal{X}$ is the real projective space $\mathbb{RP}^n$. We set $\mathcal{L} = \mathcal{O}(1) \to \mathbb{CP}^n$ as the *hyperplane line bundle*: that is, the dual of the tautological line bundle

$$\mathcal{O}(-1) = \{(\zeta, x) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \mid \zeta \in x\} \longrightarrow \mathbb{CP}^n.$$

Recall that ample line bundles on $\mathbb{CP}^n$ are of the form $\mathcal{O}(d) = (\mathcal{O}(1))^{\otimes d}$ with $d \in \mathbb{N}^*$ (see [16, Section 1.1]). The complex conjugation and the usual Hermitian inner product of $\mathbb{C}^{n+1}$ induce compatible real and metric structures on $\mathcal{O}(-1)$, hence on $\mathcal{O}(1)$ by duality. The resulting Hermitian metric on $\mathcal{O}(1)$ is positive and its curvature is the Fubini-Study Kähler form on $\mathbb{CP}^n$. With our choice of normalization, the induced Riemannian metric is the quotient of the Euclidean metric on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Finally, we choose $\mathcal{E}$ to be the rank r trivial bundle $\mathbb{C}^r \times \mathbb{CP}^n \to \mathbb{CP}^n$, with the compatible real and metric structures inherited from the standard ones in $\mathbb{C}^r$.

Notation 6.1. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. We denote its *length* by

$$|\alpha| = \alpha_0 + \cdots + \alpha_n.$$

We also define $X^\alpha = X_0^{\alpha_0} \dots X_n^{\alpha_n}$ and $\alpha! = \alpha_0! \cdots \alpha_n!$. Finally, if $|\alpha| = d$, we denote by $\binom{d}{\alpha}$ the multinomial coefficient $d!/\alpha!$.

It is well known (cf. [6, 7, 17, 22]) that $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ is the space $(\mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n])^r$ of tuples of real homogeneous polynomials of degree d in $n+1$ variables. The r terms $\mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ in $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ are pairwise orthogonal for the inner product (2.2). Besides, in restriction to one of these terms, (2.2) equals

$$(P, Q) \mapsto \int_{x \in \mathbb{CP}^n} h_d(P(x), Q(x)) \, |\mathrm{d}V_{\mathbb{CP}^n}| = \frac{1}{\pi(d+n)!} \int_{z \in \mathbb{C}^{n+1}} P(z)\overline{Q(z)} e^{-\|z\|^2} \, \mathrm{d}z. \tag{6.1}$$

An orthonormal basis of $\mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ is then

$$\left(\sqrt{\frac{(d+n)!}{\pi^n d!}} \sqrt{\binom{d}{\alpha}} X^\alpha \right)_{|\alpha| = d}.$$

Hence, a standard Gaussian in $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ is a r-tuple of independent random polynomials of the form

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha, \tag{6.2}$$

where the coefficients $(a_\alpha)_{|\alpha|=d}$ are independent real standard Gaussian variables. Such a random polynomial is called a *Kostlan-Shub-Smale polynomial* (KSS for short).

6.1.2. Correlation kernel. In this section, we study the distribution of the KSS polynomial (see equation (6.2)). In the setting of 6.1.1, E_d is the Bergman kernel of $\mathbb{C}^r \otimes \mathcal{O}(d) \to \mathbb{CP}^n$. Since the first factor is trivial, we have $E_d = I_r \otimes E'_d$, where I_r is the identity of $\mathbb{C}^r$ and E'_d is the Bergman kernel of $\mathcal{O}(d) \to \mathbb{CP}^n$. Note that E'_d is the correlation kernel of the field s'_d defined by one KSS polynomial, seen as a random section of $\mathcal{O}(d)$. By equation (6.2) we have

$$E'_d(x,y) = \mathbb{E}[s'_d(x) \otimes s'_d(y)^*] = \frac{(d+n)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} X^\alpha(x) \otimes X^\alpha(y)^*.$$

Note that (6.1) is invariant under the action of the orthogonal group $O_{n+1}(\mathbb{R})$ on the right. Hence, the distribution of KSS polynomials (6.2) and E'_d is equivariant under this action. Since $O_{n+1}(\mathbb{R})$ acts transitively on the couples of points of $\mathbb{RP}^n$ at a given distance, $E'_d(x,y)$ only depends on the geodesic distance $\rho_g(x,y)$, and the same holds for derivatives. Loosely speaking, this implies the following, where derivatives are computed with respect to the Chern connection:

(1) The variance of $s'_d(x)$ does not depend on $x \in \mathbb{RP}^n$.
(2) For all $x \in \mathbb{RP}^n$, $s'_d(x)$ and $\nabla^d_x s'_d$ are independent.
(3) If $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is any orthonormal basis of $T_x\mathbb{RP}^n$, then

$$\frac{\partial s'_d}{\partial x_i}(x) \text{ and } \frac{\partial s'_d}{\partial x_j}(x) \quad \text{are independent of } i \neq j.$$

Moreover, the variance of $(\partial s_d/\partial x_i)(x)$ does not depend on i, nor on our choice of orthonormal basis, nor on $x \in \mathbb{RP}^n$.

These properties are very specific of the case of KSS polynomials. They will be useful in Section 6.3, to compute the Wiener-Itô expansion of $\langle |\mathrm{d}V_d|, \phi\rangle$. We do not give more details here, since properties (1), (2), and (3) can also be deduced from the expression of E'_d in local coordinates that we derive below.

6.1.3. Local expression of the kernel. Let $x \in \mathbb{RP}^n$. We want to compute the expression of E'_d in some good coordinates around x. We could use the real normal trivialization, but the computations would be cumbersome. Instead, we use a slightly different trivialization. Since E'_d is equivariant under the action $O_{n+1}(\mathbb{R})$, we can assume that $x = [1 : 0 : \cdots : 0]$.

We have a chart $\psi_x : (z_1, \dots, z_n) \longmapsto [1 : z_1 : \cdots : z_n]$ from $\mathbb{R}^n$ to $B_{\mathbb{RP}^n}(x, \pi/2)$. We trivialize $\mathcal{O}(d)$ over $B_{\mathbb{RP}^n}(x, \pi/2)$ by identifying each fiber with $\mathcal{O}(d)_x$, by parallel transport with respect to the Chern connection ∇^d along curves of the form $t \mapsto \psi_x(tz)$ with $z \in \mathbb{R}^n$. Thanks to this trivialization, we can consider E'_d as a map taking values in $\mathbb{R}$. Recall that we defined a scaled version e_d of the Bergman kernel E_d by equation (3.1). The following is related without being an exact analogue. For all $w, z \in \mathbb{R}^n$, we set

$$\xi_d(w,z) = \frac{\pi^n d!}{(d+n)!} E'_d(\psi_x(w), \psi_x(z)). \tag{6.3}$$

A computation in local coordinates yields the following lemma. The Chern connection ∇^d coincides at the origin with the usual differential in our trivialization. Hence, taking the values at $(0,0)$ of the following expressions proves that s'_d satisfies properties (1), (2) and (3) (cf. Subsection 6.1.2).

Lemma 6.2. *Let $d \in \mathbb{N}^*$ and let $i, j \in \{1, \dots, n\}$. Then, for all $w, z \in \mathbb{R}^n$ we have*

$$\begin{aligned}
\xi_d(w,z) &= \left(\frac{1 + \langle w, z\rangle}{\sqrt{1 + \|w\|^2}\sqrt{1 + \|z\|^2}}\right)^d,\\
\partial_{x_i}\xi_d(w,z) &= d\xi_d(w,z)\left(\frac{z_i}{1 + \langle w,z\rangle} - \frac{w_i}{1 + \|w\|^2}\right),\\
\partial_{y_j}\xi_d(w,z) &= d\xi_d(w,z)\left(\frac{w_j}{1 + \langle w,z\rangle} - \frac{z_j}{1 + \|z\|^2}\right),\\
\partial_{x_i}\partial_{y_j}\xi_d(w,z) &= \xi_d(w,z)\Bigg(\frac{d\delta_{ij}}{1 + \langle w,z\rangle} - \frac{d^2 w_i w_j}{(1 + \langle w,z\rangle)(1 + \|w\|^2)}\\
&\qquad - \frac{d^2 z_i z_j}{(1 + \langle w,z\rangle)(1 + \|z\|^2)}\\
&\qquad + \frac{d^2 w_i z_j}{(1 + \|w\|^2)(1 + \|z\|^2)} + \frac{(d^2 - d) z_i w_j}{(1 + \langle w,z\rangle)^2}\Bigg),
\end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

6.2. Hermite polynomials and Wiener chaos. In the setting of KSS polynomials, we consider $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d)) = (\mathbb{R}_d^{\hom}[X_0, \dots, X_n])^r$, equipped with the inner product (6.1). For simplicty, in this section and the following, we denote by V_d this Euclidean space and by $\mathrm{d}\nu_d$ its standard Gaussian measure. With these notation, $(V_d, \mathrm{d}\nu_d)$ is our probability space and we denote by $L^1(\mathrm{d}\nu_d)$ (respectively, $L^2(\mathrm{d}\nu_d)$) the space of integrable (respectively, square integrable) random variables on this space. Theorem 1.6 shows that for d large enough, for all $\phi \in C^0(\mathbb{RP}^n)$, $\langle |\mathrm{d}V_d|, \phi\rangle \in L^2(\mathrm{d}\nu_d)$. The proof given in Section 5 shows that this is true for any $d \geqslant \max(d_0, d_1, d_2, d_3)$; in this framework it is true for any $d \in \mathbb{N}^*$. The idea of this section is to find a nice orthogonal decomposition of $L^2(\mathrm{d}\nu_d)$. In 6.3 we will study $\langle |\mathrm{d}V_d|, \phi \rangle$ thanks to this decomposition. (These techniques were already used in a similar context in [2, 11, 12, 25], e.g. See [28] for background on the following material.)

Definition 6.3. For all $k \in \mathbb{N}$, we denote by H_k the k-th *Hermite polynomial*. These polynomials are defined recursively by $H_0 = 1$, $H_1 = X$, and, for all $k \in \mathbb{N}^*$, $H_{k+1}(X) = XH_k(X) - kH_{k-1}(X)$.

Remark 6.4. Equivalently, one can define H_k by $H_0 = 1$ and, for all $k \in \mathbb{N}$, by $H_{k+1} = XH_k - H_k'$.

Lemma 6.5. *Let $k \in \mathbb{N}$; then, H_k is a polynomial of degree k which is even if k is even and odd if k is odd. Moreover,*

$$H_{2k}(0) = (-1)^k \frac{(2k)!}{2^k k!} \quad \text{and} \quad H_{2k+1}(0) = 0.$$

Proof. This is proved by induction, by using the recursive definition of H_k. $\square$

Let us denote by $\mathrm{d}\mu_N$ the standard Gaussian measure on $\mathbb{R}^N$. We also denote by $L^2(\mathrm{d}\mu_N)$ the space of square integrable functions with respect to $\mathrm{d}\mu_N$. Recall that the family $((1/\sqrt{k!})H_k)_{k \in \mathbb{N}}$ is a Hilbert basis of $L^2(\mathrm{d}\mu_1)$ (see Proposition 1.4.2 in [28]). Similarly, in dimension N, the family

$$\left\{ \prod_{i=1}^{N} \frac{1}{\sqrt{\alpha_i!}} H_{\alpha_i}(X_i) \mid \alpha \in \mathbb{N}^N \right\}$$

is a Hilbert basis of $L^2(\mathrm{d}\mu_N)$. The result in dimension 1 shows that this family is orthonormal. Then, one only needs to check that the space of polynomials in N variables is dense in $L^2(\mathrm{d}\mu_N)$. For $N = 1$ this is proved in Proposition 1.1.5 of [28], and the same proof works in any dimension.

As in Section 5, we denote by s_d a generic element of $(V_d, \mathrm{d}\nu_d)$, which we think of as a standard Gaussian vector in V_d. Let $\eta \in V_d^*$; then, $\eta(s_d) \in L^2(\mathrm{d}\nu_d)$ is a real centered Gaussian variable. Moreover, for any $\eta, \eta' \in V_d^*$, we have $\mathbb{E}[\eta(s_d)\eta'(s_d)] = \langle \eta, \eta' \rangle$. Thus, V_d^* is canonically isometric to a subspace of $L^2(\mathrm{d}\nu_d)$, via $\eta \mapsto \eta(s_d)$. From now on, we identify V_d^* with its image, so that $V_d^* \subset L^2(\mathrm{d}\nu_d)$ is a centered Gaussian Hilbert space.

Definition 6.6. Let $(\eta_1, \ldots, \eta_{N_d})$ denote an orthonormal basis of V_d^*; that is, the $(\eta_i(s_d))_{i \in \{1, \ldots, N_d\}}$ are independent real standard Gaussian variables. For all $q \in \mathbb{N}$, we define $C_d[q]$, the *q-th Wiener chaos* of the field s_d, as the subspace of $L^2(\mathrm{d}\nu_d)$ spanned by the orthogonal family

$$\left\{ \prod_{i=1}^{N_d} H_{\alpha_i}(\eta_i) \mid \alpha \in \mathbb{N}^{N_d},\ |\alpha| = q \right\}.$$

Remarks 6.7.

- $C_d[0]$ is the space of constant random variables in $L^2(\mathrm{d}\nu_d)$ and $C_d[1] = V_d^*$.
- We do not need to take the closure in the definition of $C_d[q]$ since it is finite dimensional.

Lemma 6.8. *The Wiener chaoses $(C_d[q])_{q\in\mathbb{N}}$ of s_d do not depend on the choice of the orthonormal basis $(\eta_1,\dots,\eta_{N_d})$ appearing in Definition 6.6.*

Proof. Let $(\eta_1,\dots,\eta_{N_d})$ and $(\eta'_1,\dots,\eta'_{N_d})$ be two orthonormal basis of V_d^*. There is an orthogonal transformation U of V_d^* such that, for all $i\in\{1,\dots,N_d\}$, we have $\eta'_i = U(\eta_i)$. As the situation is symmetric, we only have to prove (for any $\beta\in\mathbb{N}^{N_d}$ such that $|\beta| = q$) that $\prod_{i=1}^{N_d} H_{\beta_i}(\eta_i)$ is a linear combination of elements of the family: $\{\prod_{i=1}^{N_d} H_{\alpha_i}(U(\eta_i)) \mid \alpha\in\mathbb{N}^{N_d},\ |\alpha| = q\}$. Dropping the dependence on d, this amounts to proving that if $X = (X_1,\dots,X_N)\in\mathbb{R}^N$ and $U\in O_N(\mathbb{R})$, then, for all $\beta\in\mathbb{N}^N$ such that $|\beta| = q$, $\prod_{i=1}^N H_{\beta_i}(X_i)$ is a linear combination of the $(\prod_{i=1}^N H_{\alpha_i}(U(X_i)))_{|\alpha|=q}$.

By [28, Proposition 1.4.2], we have

$$\forall\, t\in\mathbb{R}^n,\qquad \sum_{\alpha\in\mathbb{N}^N}\Big(\prod_{i=1}^N H_{\alpha_i}(X_i)\Big)\frac{t^\alpha}{\alpha!} = \exp\left(\langle t, X\rangle - \frac{\|t\|^2}{2}\right),$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^N$ and $\|\cdot\|$ the associated norm. As the righthand side is invariant under orthogonal transformation, we have

$$\forall\, t\in\mathbb{R}^n,\qquad \sum_{\alpha\in\mathbb{N}^N}\Big(\prod_{i=1}^N H_{\alpha_i}(U(X_i))\Big)\frac{(U(t))^\alpha}{\alpha!} = \sum_{\alpha\in\mathbb{N}^N}\Big(\prod_{i=1}^N H_{\alpha_i}(X_i)\Big)\frac{t^\alpha}{\alpha!}.$$

Now, note that the components of $U(t)$ are homogeneous polynomials of degree 1 in $(t_1,\dots,t_N)$. Hence, $(U(t))^\alpha$ can only contribute terms of degree $|\alpha|$ to the sum.

We conclude by identifying the coefficients of these power series of the variable t. □

Lemma 6.9. *For all $d\in\mathbb{N}^*$, $\bigoplus_{q\in\mathbb{N}} C_d[q]$ is dense in $L^2(\mathrm{d}\nu_d)$. Moreover, the terms of this direct sum are pairwise orthogonal.*

Proof. As the family $(\prod_{i=1}^{N_d} H_{\alpha_i}(X_i))_{\alpha\in\mathbb{N}^{N_d}}$ is orthogonal, the $(C_d[q])_{q\in\mathbb{N}}$ are pairwise orthogonal by definition. Let $(s_{1,d},\dots,s_{N_d,d})$ be an orthonormal basis of V_d. We have $s_d = \sum a_i s_{i,d}$, where the a_i are independent $\mathcal{N}(1)$. For any $i\in\{1,\dots,N_d\}$, let $\eta_i = \langle\cdot, s_{i,d}\rangle$, so that $\eta_i(s_d) = a_i$. Then, for any $q\in\mathbb{N}$, $C_d[q]$ is spanned by the random variables $(\prod_{i=1}^{N_d} H_{\alpha_i}(a_i))_{|\alpha|=q}$.

Any square integrable functional of s_d can be written as $F(a_1,\dots,a_{N_d})$, with $F\in L^2(\mathrm{d}\mu_{N_d})$. Since the span of $\{\prod_{i=1}^{N_d} H_{\alpha_i}(X_i) \mid \alpha\in\mathbb{N}^{N_d}\}$ is dense in $L^2(\mathrm{d}\mu_{N_d})$, the conclusion follows. □

Notation 6.10. Let $d \in \mathbb{N}^*$ and let $A \in L^2(\mathrm{d}\nu_d)$. For any $q \in \mathbb{N}$, we denote by $A[q]$ the *q-th chaotic component* of A, that is, its projection onto $C_d[q]$. Then, we have $A = \sum_{q\in\mathbb{N}} A[q]$ in $L^2(\mathrm{d}\nu_d)$.

By definition, $A[0] = \mathbb{E}[A]$. Moreover, as the $C_d[q]$ are pairwise orthogonal, we have $\mathbb{E}[A[q]] = 0$ for any $q \in \mathbb{N}^*$, and $\mathrm{Var}(A) = \sum_{q\in\mathbb{N}^*} \mathrm{Var}(A[q])$.

6.3. Wiener-Itô expansion of the linear statistics. Recall that we consider a standard Gaussian section $s_d \in V_d = (\mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n])^r$ and that $|\mathrm{d}V_d|$ denotes the Riemannian measure of integration over its real zero set. Let us fix $d \in \mathbb{N}^*$ and $\phi \in C^0(\mathbb{RP}^n)$. By Theorem 1.6, $\langle |\mathrm{d}V_d|, \phi\rangle \in L^2(\mathrm{d}\nu_d)$. The goal of this section is to compute the chaotic expansion of these variables. For all $q \in \mathbb{N}$, we denote $\langle |\mathrm{d}V_d|\,[q], \phi\rangle$ for $\langle |\mathrm{d}V_d|, \phi\rangle)[q]$.

Since $\langle |\mathrm{d}V_d|, \phi\rangle \in L^2(\mathrm{d}\nu_d)$, for any $A \in L^2(\mathrm{d}\nu_d)$ we have

$$(A\langle |\mathrm{d}V_d|, \phi\rangle) \in L^1(\mathrm{d}\nu_d),$$

and

$$\mathbb{E}[A\langle |\mathrm{d}V_d|, \phi\rangle] = \mathbb{E}\Big[\int_{x\in Z_d} \phi(x) A(s_d)\,|\mathrm{d}V_d|\Big].$$

Even if A depends on s_d, we can apply a Kac-Rice formula (cf. Theorem 5.3 of [22]). Thus, we have

$$\mathbb{E}[A\langle |\mathrm{d}V_d|, \phi\rangle] = (2\pi)^{-r/2}\int_{x\in\mathbb{RP}^n} \frac{\phi(x)}{|\det^{\perp}(\mathrm{ev}_x^d)|} \times \mathbb{E}\big[A|\det^{\perp}(\nabla_x^d s_d)| : s_d(x) = 0\big]\,|\mathrm{d}V_{\mathbb{RP}^n}|.$$

Recall that s_d is a tuple of independent KSS polynomials, and that $E_d = I_r \otimes E'_d$, where E'_d is the correlation kernel of one KSS polynomial. By equation (6.3) and Lemma 6.2, we have

$$|\det^{\perp}(\mathrm{ev}_x^d)| = \det(E_d(x,x))^{1/2} = \det(E'_d(x,x))^{r/2} = \left(\frac{(d+n)!}{\pi^n d!}\right)^{r/2}.$$

Denoting

$$\left(d\frac{(d+n)!}{\pi^n d!}\right)^{-1/2} \nabla_x^d s_d \quad \text{by } L_d(x),$$
$$\left(\frac{(d+n)!}{\pi^n d!}\right)^{-1/2} s_d(x) \quad \text{by } t_d(x),$$

we get

$$\mathbb{E}[A\langle |\mathrm{d}V_d|, \phi\rangle] = \left(\frac{d}{2\pi}\right)^{r/2}\int_{x\in\mathbb{RP}^n} \phi(x) \times \mathbb{E}\big[A|\det^{\perp}(L_d(x))| : t_d(x) = 0\big]\,|\mathrm{d}V_{\mathbb{RP}^n}|. \tag{6.4}$$

Let $x \in \mathbb{R}\mathbb{P}^n$; without loss of generality, we can assume that the coordinates on $\mathbb{R}^{n+1}$ are such that $x = [1 : 0 : \cdots : 0]$. Let $\zeta_0(x)$ be one of the two unit vectors in $\mathbb{R}\mathcal{O}(d)_x$, the other one being $-\zeta_0(x)$. This gives an isomorphism between $(\mathbb{R} \oplus T_x^*(\mathbb{R}\mathbb{P}^n)) \otimes \mathcal{O}(d)_x$ and $\mathbb{R} \oplus T_x^*(\mathbb{R}\mathbb{P}^n)$, so that we can consider $(t_d(x), L_d(x))$ as an element of $\mathbb{R}^r \oplus (T_x^*\mathbb{R}\mathbb{P}^n)^r$. We denote by $(t_d^{(1)}(x), \ldots, t_d^{(r)}(x))$ the components of $t_d(x)$, and by $(L_d^{(1)}(x), \ldots, L_d^{(r)}(x))$ those of $L_d(x)$. The couples $(t_d^{(i)}(x), L_d^{(i)}(x))$ are centered Gaussian vectors in $\mathbb{R} \oplus T_x^*\mathbb{R}\mathbb{P}^n$ that are independent from one another. Moreover, by Lemma 6.2, for all $i \in \{1, \ldots, r\}$, the variance operator of $(t_d^{(i)}(x), L_d^{(i)}(x))$ is Id.

Let us choose any orthonormal basis of $T_x^*\mathbb{R}\mathbb{P}^n$, and denote the coordinates of $L_d^{(i)}(x)$ in this basis by $(L_d^{i1}(x), \ldots, L_d^{in}(x))$, so that

$$\left(L_d^{ij}(x)\right)_{\substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant n}} \quad \text{is the matrix of } L_d(x).$$

Then,

$$\{t_d^{(i)}(x) \mid 1 \leqslant i \leqslant r\} \sqcup \{L_d^{ij}(x) \mid 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n\}$$

is a family of independent real standard Gaussian variables in $L^2(\mathrm{d}\nu_d)$, and we can complete it into an orthonormal basis of $C_d[1]$. We therefore denote by $\{S_d^{(i)}(x) \mid r(n+1) < i \leqslant N_d\}$ the last elements of such a basis. Below, we work in the Hilbert basis of $L^2(\mathrm{d}\nu_d)$ obtained by considering Hermite polynomials in these variables.

Remark 6.11. We just used the fact that our random field satisfies properties (1), (2), and (3) of Subsection 6.1.2. This is what makes this computation specific to the case of KSS polynomials.

Notation 6.12. Let $\alpha \in \mathbb{N}^r$, $\beta \in \mathbb{N}^r \times \mathbb{N}^n$ and $\gamma \in \mathbb{N}^{N_d - r(n+1)}$. We will use the following notation:

$$\begin{aligned}
H_\alpha(t_d(x)) &= \prod_{i=1}^{r} H_{\alpha_i}(t_d^{(i)}(x)),\\
\tilde{H}_\beta(L_d(x)) &= \prod_{1\leqslant i\leqslant r,\, 1\leqslant j\leqslant n} H_{\beta_{ij}}(L_d^{ij}(x)),\\
\hat{H}_\gamma(S_d(x)) &= \prod_{i=r(n+1)+1}^{N_d} H_{\gamma_i}(S_d^{(i)}(x)).
\end{aligned}$$

We first expand $|\det^\perp(L_d(x))|$ in $L^2(\mathrm{d}\nu_d)$. Since $|\det^\perp(L_d(x))|$ only depends on the variables $\{L_d^{ij}(x) \mid 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n\}$, we have

$$|\det{}^\perp(L_d(x))| = \sum_{\beta \in \mathbb{N}^r \times \mathbb{N}^n} B_\beta \frac{\tilde{H}_\beta(L_d(x))}{\sqrt{\beta!}},$$

where

$$B_\beta = \frac{1}{\sqrt{\beta!}}\mathbb{E}[|\det{}^\perp(L_d(x))|\tilde{H}_\beta(L_d(x))] \quad \forall\, \beta \in \mathbb{N}^r \times \mathbb{N}^n.$$

The coefficient B_β only depends on the distribution of $L_d(x)$, which is a standard Gaussian for all $x \in \mathbb{RP}^n$. Hence, B_β is independent of x. These coefficients have several symmetries. Note that $|\det{}^\perp((L_d^{ij}(x))_{i,j})|$ is invariant under the following operations:

- multiplying a whole column or a whole row by -1
- permuting the rows or permuting the columns.

Since the Hermite polynomials of odd degrees are odd (cf. Lemma 6.5), the first point shows that $B_\beta = 0$ whenever there exists $i \in \{1, \dots, r\}$ such that $\sum_{j=1}^n \beta_{ij}$ is odd or there exists $j \in \{1, \dots, n\}$ such that $\sum_{i=1}^r \beta_{ij}$ is odd. We denote by I the set of multi-indices $\beta \in \mathbb{N}^r \times \mathbb{N}^n$ such that, for all $i \in \{1, \dots, r\}$, $\sum_{j=1}^n \beta_{ij}$ is even, and, for all $j \in \{1, \dots, n\}$, $\sum_{i=1}^r \beta_{ij}$ is even.

If $|\beta| = 2$, then the only way for β to belong to I is that there exist (i, j) such that $\beta_{ij} = 2$, the other components of β being zero. The second point above shows that, in this case, the value of B_β does not depend on the index (i, j) such that $\beta_{ij} = 2$.

Notation 6.13. Let B_2 denote the common value of the B_β for $\beta \in I$ such that $|\beta| = 2$.

We can also expand $A \in L^2(\mathrm{d}\nu_d)$ as

$$A = \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma}(x)\frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}}\frac{\tilde{H}_\beta(L_d(x))}{\sqrt{\beta!}}\frac{\hat{H}_\gamma(S_d(x))}{\sqrt{\gamma!}},$$

where

$$A_{\alpha,\beta,\gamma}(x) = \mathbb{E}\left[A\frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}}\frac{\tilde{H}_\beta(L_d(x))}{\sqrt{\beta!}}\frac{\hat{H}_\gamma(S_d(x))}{\sqrt{\gamma!}}\right].$$

Then, using the orthonormality properties of the Hermite polynomials, we get

$$\mathbb{E}[A|\det{}^\perp(L_d(x))| : t_d(x) = 0] = \sum_{\alpha,\beta} A_{\alpha,\beta,0}(x) B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}}, \tag{6.5}$$

where the sum runs over multi-indices such that $\alpha \in 2\mathbb{N}^r$ (see Lemma 6.5), and $\beta \in I$. Then, by equations (6.4) and (6.5), for any $A \in C_d[q]$, we have

$$\mathbb{E}[A\langle|\mathrm{d}V_d|, \phi\rangle] = \left(\frac{d}{2\pi}\right)^{r/2} \times \sum_{\substack{\alpha\in 2\mathbb{N}^r,\, \beta\in I \\ |\alpha|+|\beta|=q}} B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}}\mathbb{E}\left[A\int_{x\in\mathbb{RP}^n}\phi(x)\frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}}\frac{\tilde{H}_\beta(L_d(x))}{\sqrt{\beta!}}|\mathrm{d}V_{\mathbb{RP}^n}|\right].$$

We have proved the following proposition.

Proposition 6.14. *For all $d \in \mathbb{N}^*$, for all $\phi \in C^0(\mathbb{RP}^n)$, for all $q \in \mathbb{N}$, we have* $\langle |\mathrm{d}V_d|\,[2q+1], \phi\rangle = 0$, *and*

$$\langle |\mathrm{d}V_d|\,[2q], \phi\rangle = \left(\frac{d}{2\pi}\right)^{r/2} \times \int_{x\in\mathbb{RP}^n} \phi(x) \sum_{\substack{\alpha\in 2\mathbb{N}^r,\, \beta\in I\\ |\alpha|+|\beta|=2q}} B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}}\frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}}\frac{\tilde{H}_\beta(L_d(x))}{\sqrt{\beta!}}\,|\mathrm{d}V_{\mathbb{RP}^n}|. \tag{6.6}$$

Remarks 6.15.

- Recall that the values of the $t_d^{(i)}(x)$ and $L_d^{ij}(x)$ depend on the choice of the unit vector $\zeta_0(x)$, which we used to trivialize $\mathcal{O}(d)_x$. The only other choice of such a unit vector is $-\zeta_0(x)$. Changing $\zeta_0(x)$ to $-\zeta_0(x)$ changes $t_d^{(i)}(x)$ to $-t_d^{(i)}(x)$ and $L_d^{ij}(x)$ to $-L_d^{ij}(x)$. Since we only consider multi-indices (α,β) such that $|\alpha|+|\beta|$ is even, the monomials appearing in $H_\alpha(t_d(x))\tilde{H}_\beta(L_d(x))$ with a non-zero coefficient have even total degree. Hence, the value of $H_\alpha(t_d(x))\tilde{H}_\beta(L_d(x))$ does not depend on the choice of $\zeta_0(x)$.
- Since $\sum_{\beta\in I,\,|\beta|=p}(B_\beta/\sqrt{\beta!})\tilde{H}_\beta(L_d(x))$ is the p-th chaotic component of $|\det^\perp(L_d(x))|$, it does not depend on our choice of an orthonormal basis of $T_x^*\mathbb{RP}^n$. Hence, neither does the value of the sum on the righthand side of equation (6.6), for any given $x\in\mathbb{RP}^n$.
- By [21, Lemma A.14],

$$B_0 = \mathbb{E}[|\det^\perp(L_x(d))|] = (2\pi)^r\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}.$$

  Then, Proposition 6.14 for $q=0$ shows that, in the setting of KSS polynomials, for all $\phi\in C^0(\mathbb{RP}^n)$,

$$\mathbb{E}[\langle|\mathrm{d}V_d|,\phi\rangle] = d^{r/2}\left(\int_{\mathbb{RP}^n}\phi\,|\mathrm{d}V_{\mathbb{RP}^n}|\right)\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}.$$

  That is, in this case, the error term in Theorem 1.2 is zero for any $d\in\mathbb{N}^*$.

Let us conclude this section by writing $\langle|\mathrm{d}V_d|\,[2],\phi\rangle$ in a more explicit way.

Lemma 6.16. *For all $d\in\mathbb{N}^*$, for all $\phi\in C^0(\mathbb{RP}^n)$,*

$$\langle|\mathrm{d}V_d|\,[2],\phi\rangle = d^{r/2}\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{2n\,\mathrm{Vol}(\mathbb{S}^n)} \times \int_{x\in\mathbb{RP}^n}\phi(x)(\|L_d(x)\|^2 - n\|t_d(x)\|^2)\,|\mathrm{d}V_{\mathbb{RP}^n}|.$$

Proof. By Proposition 6.14 and Lemma 6.5, we have

$$(6.7)\quad \langle |\mathrm{d}V_d|[2], \phi\rangle = \left(\frac{d}{2\pi}\right)^{r/2} \times \int_{x\in\mathbb{RP}^n} \phi(x)\left(-\frac{B_0}{2}(\|t_d(x)\|^2 - r) + \frac{B_2}{\sqrt{2}}(\|L_d(x)\|^2 - nr)\right)|\mathrm{d}V_{\mathbb{RP}^n}|,$$

where B_2 is defined by Notation 6.13. Since $H_2 = X^2 - 1$, we have

$$\begin{aligned} n\sqrt{2}B_2 &= \sum_{j=1}^{n} \mathbb{E}[|\det{}^{\perp}(L_d(x))|H_2(L_d^{1j}(x))] \\ &= \mathbb{E}[|\det{}^{\perp}(L_d(x))|\,\|L_d^{(1)}(x)^2\|] - nB_0. \end{aligned}$$

It was proved in [21, Appendix B] that $|\det^{\perp}(L_d(x))|$ is distributed as

$$\|L_d^{(1)}(x)\|\,\|Z_{n-1}\|\ldots\|Z_{n-r+1}\|,$$

where $(L_d^{(1)}(x), Z_{n-1}, \ldots, Z_{n-r+1})$ are globally independent and Z_p is a standard Gaussian vector in $\mathbb{R}^p$, for all $p \in \{n-r+1, \ldots, n-1\}$. Since $L_d^{(1)}(x) \sim \mathcal{N}(\mathrm{Id})$ in a Euclidean space of dimension n, we have

$$\begin{aligned} B_0 &= \mathbb{E}[|\det{}^{\perp}(L_d(x))|] = \mathbb{E}[\|L_d^{(1)}(x)\|]\prod_{p=n-r+1}^{n-1}\mathbb{E}[\|Z_p\|] \\ &= (2\pi)^{r/2}\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}, \\ B_2 &= \frac{1}{n\sqrt{2}}\mathbb{E}[\|L_d^{(1)}(x)\|^3]\prod_{p=n-r+1}^{n-1}\mathbb{E}[\|Z_p\|] - \frac{B_0}{\sqrt{2}} \\ &= \frac{B_0}{\sqrt{2}}\left(\frac{2\pi}{n}\frac{\mathrm{Vol}(\mathbb{S}^n)}{\mathrm{Vol}(\mathbb{S}^{n+2})} - 1\right) = \frac{B_0}{n\sqrt{2}}. \end{aligned}$$

We plug these relations in equation (6.7), and this yields the result. □

6.4. Conclusion of the proof. In this section, we finally prove Theorem 1.8. The key point is the following.

Lemma 6.17. *Let Z_d be the common zero set of r independent Kostlan-Shub-Smale polynomials in $\mathbb{RP}^n$; then, we have the following as d goes to infinity:*

$$\mathrm{Var}(\mathrm{Vol}(Z_d)[2]) \sim d^{r-n/2} r\left(1 + \frac{2}{n}\right)\pi^{n/2}\frac{\mathrm{Vol}(\mathbb{S}^{n-r})^2}{16\,\mathrm{Vol}(\mathbb{S}^n)}.$$

Let us prove that this lemma implies Theorem 1.8.

Proof of Theorem 1.8. Let us consider the common zero set Z_d of r independent KSS polynomials in $\mathbb{RP}^n$, and denote by $|\mathrm{d}V_d|$ the Riemannian volume measure on Z_d. Let $\mathbf{1}$ be the unit constant function on $\mathbb{RP}^n$. We thus have $\langle |\mathrm{d}V_d|, \mathbf{1}\rangle = \mathrm{Vol}(Z_d)$ and, by Theorem 1.6,

$$d^{-r+n/2}\,\mathrm{Var}(\mathrm{Vol}(Z_d)) = \mathrm{Vol}(\mathbb{RP}^n)\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r}\mathcal{I}_{n,r} + \delta_{rn}\frac{2}{\mathrm{Vol}(\mathbb{S}^n)}\right) + o(1).$$

On the other hand, as we explained at the end of Subsection 6.2,

$$\begin{aligned} d^{-r+n/2}\,\mathrm{Var}(\mathrm{Vol}(Z_d)) &= d^{-r+n/2}\sum_{q\in\mathbb{N}^*}\mathrm{Var}(\mathrm{Vol}(Z_d)[q]) \\ &\geqslant d^{-r+n/2}\,\mathrm{Var}(\mathrm{Vol}(Z_d)[2]). \end{aligned}$$

By Lemma 6.17, we get

$$\begin{aligned} &\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r}\mathcal{I}_{n,r} + \delta_{rn}\frac{2}{\mathrm{Vol}(\mathbb{S}^n)}\right) \\ &\qquad \geqslant \frac{r}{8}\left(1+\frac{2}{n}\right)\pi^{n/2}\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2 > 0. \end{aligned}$$ □

Remark 6.18. In [11], Dalmao proved that for $n = r = 1$, one obtains $\mathrm{Var}(\mathrm{Vol}(Z_d)) \sim \sigma^2\sqrt{d}$ with $\sigma^2 \simeq 0.57\ldots$. What we just said shows that $\sigma^2 = 1 + \mathcal{I}_{1,1}$, and the lower bound we get for this term in the proof of Theorem 1.8 is $3/(8\sqrt{\pi}) \simeq 0.21\ldots$. Thus, asymptotically, chaotic components of order greater than 4 must contribute to the leading term of $\mathrm{Var}(\mathrm{Vol}(Z_d))$.

We conclude this section by the proof of Lemma 6.17.

Proof of Lemma 6.17. Recall that $|\mathrm{d}V_d|$ is the Riemannian volume measure on Z_d, and that $\mathbf{1}$ is the unit constant function on $\mathbb{RP}^n$. By Lemma 6.16, we have

$$\begin{aligned} \mathrm{Vol}(Z_d)[2] &= \langle |\mathrm{d}V_d|\,[2], \mathbf{1}\rangle \\ &= d^{r/2}\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{2n\,\mathrm{Vol}(\mathbb{S}^n)}\int_{x\in\mathbb{RP}^n}\left(\|L_d(x)\|^2 - n\|t_d(x)\|^2\right)|\mathrm{d}V_{\mathbb{RP}^n}|. \end{aligned}$$

Since this is a centered variable, its variance equals

$$\begin{aligned} &d^r\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{2n\,\mathrm{Vol}(\mathbb{S}^n)}\right)^2 \\ &\quad\times\int_{(x,y)\in(\mathbb{RP}^n)^2}\mathbb{E}\left[\left(\|L_d(x)\|^2 - n\|t_d(x)\|^2\right)\left(\|L_d(y)\|^2 - n\|t_d(y)\|^2\right)\right]|\mathrm{d}V_{\mathbb{RP}^n}|^2. \end{aligned}$$

Using the invariance of the distribution of s_d under isometries, we get that

$$\mathrm{Var}(\mathrm{Vol}(Z_d)[2]) = d^r \frac{\mathrm{Vol}(\mathbb{S}^{n-r})^2}{8n^2\,\mathrm{Vol}(\mathbb{S}^n))} \mathcal{J}_{n,r}(d), \tag{6.8}$$

where, setting $x_0 = [1 : 0 : \cdots : 0]$,

$$\begin{aligned}&\mathcal{J}_{n,r}(d)\\ &\quad = \int_{y \in \mathbb{RP}^n} \mathbb{E}[(\|L_d(x_0)\|^2 - n\|t_d(x_0)\|^2)(\|L_d(y)\|^2 - n\|t_d(y)\|^2)]\,|\mathrm{d}V_{\mathbb{RP}^n}|.\end{aligned}$$

Since $B_{\mathbb{RP}^n}(x_0, \pi/2) = \{[1 : z_1 : \cdots : z_n] \in \mathbb{RP}^n \mid z \in \mathbb{R}^n\}$ has full measure in $\mathbb{RP}^n$, we can restrict the above integral to this ball and use the local coordinates introduced in Subsection 6.1.3. These coordinates are centered at x_0. Moreover, note that the density of $|\mathrm{d}V_{\mathbb{RP}^n}|$ with respect to the Lebesgue measure in this chart is $z \mapsto (1 + \|z\|^2)^{-(n+1)/2}$ (cf. [16, p. 30]). Moreover, by a change of variable $y = [1 : z_1 : \cdots : z_n]$, we have

$$\mathcal{J}_{n,r}(d) = \int_{z \in \mathbb{R}^n} \mathcal{F}_d(z)(1 + \|z\|^2)^{-(n+1)/2}\,\mathrm{d}z, \tag{6.9}$$

where

$$\mathcal{F}_d(z) = \mathbb{E}[(\|L_d(0)\|^2 - n\|t_d(0)\|^2)(\|L_d(z)\|^2 - n\|t_d(z)\|^2)].$$

Here, we denoted $t_d(z)$ instead of $t_d([1 : z_1 : \cdots : z_n])$ and $L_d(z)$ instead of $L_d([1 : z_1 : \cdots : z_n])$.

Let us fix $z \in \mathbb{R}^n$ and compute $\mathcal{F}_d(z)$. Using once again the invariance under the action of $O_{n+1}(\mathbb{R})$ on $\mathbb{RP}^n$, we can assume that $z = (\|z\|, 0, \ldots, 0)$. Let

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

denote the basis of the tangent space of $\mathbb{RP}^n$ at $[1 : \|z\| : 0 : \cdots : 0]$ given by the partial derivatives in our chart ψ_{x_0} (see Subsection 6.1.3). This basis is orthogonal, but $\|\partial/\partial x_1\| = (1 + \|z\|^2)^{-1}$ and $\|\partial/\partial x_j\| = (1 + \|z\|^2)^{-1/2}$ for all $j \in \{2, \ldots, n\}$.

The random vectors $(t_d^{(i)}(0), t_d^{(i)}(z), L_d^{i1}(0), L_d^{i1}(z), \ldots, L_d^{in}(0), L_d^{in}(z))$ for $i \in \{1, \ldots, r\}$ are independent equidistributed centered Gaussian vectors in $\mathbb{R}^{2n+2}$. The previous relations, together with Lemma 6.2, show that their common variance matrix, by blocks of size 2×2, is

$$\begin{pmatrix} A_d(\|z\|^2) & (B_d(\|z\|^2))^{\mathrm{t}} & 0 & \cdots & 0 \\ B_d(\|z\|^2) & D_d(\|z\|^2) & 0 & \cdots & 0 \\ 0 & 0 & C_d(\|z\|^2) & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & C_d(\|z\|^2) \end{pmatrix}, \tag{6.10}$$

where, for all $t \geqslant 0$,

$$A_d(t) = \begin{pmatrix} 1 & (1+t)^{-d/2} \\ (1+t)^{-d/2} & 1 \end{pmatrix}, \tag{6.11a}$$

$$B_d(t) = \begin{pmatrix} 0 & \sqrt{dt}(1+t)^{-d/2} \\ -\sqrt{dt}(1+t)^{-d/2} & 0 \end{pmatrix}, \tag{6.11b}$$

$$C_d(t) = \begin{pmatrix} 1 & (1+t)^{(1-d)/2} \\ (1+t)^{(1-d)/2} & 1 \end{pmatrix}, \tag{6.11c}$$

$$D_d(t) = \begin{pmatrix} 1 & (1+t-dt)(1+t)^{-d/2} \\ (1+t-dt)(1+t)^{-d/2} & 1 \end{pmatrix}. \tag{6.11d}$$

Using the independence and equidistribution of the couples $(t_d^{(i)}(x), L_d^{(i)}(x))$, we have

$$\begin{aligned} \mathcal{F}_d(z) = r\Big(&\sum_{j,\ell} \mathbb{E}[(L_d^{1j}(0))^2(L_d^{1\ell}(z))^2] - n\sum_{\ell} \mathbb{E}[(t_d^{(1)}(0))^2(L_d^{1\ell}(z))^2] \\ &- n\sum_{j} \mathbb{E}[(L_d^{1j}(0))^2(t_d^{(1)}(z))^2] + n^2\mathbb{E}[(t_d^{(1)}(0))^2(t_d^{(1)}(z))^2]\Big). \end{aligned}$$

If (X,Y) is a centered Gaussian vector in $\mathbb{R}^2$ such that $\mathrm{Var}(X) = 1 = \mathrm{Var}(Y)$, then by Wick's formula (cf. [1, Lemma 11.6.1]) we have $\mathbb{E}[X^2Y^2] = 9 + 2\mathbb{E}[XY]^2$. We apply this relation to each term of the previous sum. Then, by equations (6.10) and (6.11a)–(6.11d), we have $\mathcal{F}_d(z) = 2rF_d(d\|z\|^2)$, where F_d is defined by

$$F_d(t) = \left(1+\frac{t}{d}\right)^{-d}\left(\left(1+\frac{t}{d}-t\right)^2 + (n-1)\left(1+\frac{t}{d}\right) - 2nt + n^2\right),$$

for all $t \in \mathbb{R}$. Then, by a change of variable $t = d\|z\|^2$ in equation (6.9),

$$\mathcal{I}_{n,r}(d) = d^{-n/2} r \,\mathrm{Vol}(\mathbb{S}^{n-1}) \times \int_{t=0}^{+\infty} F_d(t) t^{(n-2)/2}\left(1+\frac{t}{d}\right)^{-(n+1)/2} \mathrm{d}t. \tag{6.12}$$

Let $t \geqslant 0$; we then have

$$F_d(t)t^{(n-2)/2}\left(1+\frac{t}{d}\right)^{-(n+1)/2} \xrightarrow[d\to+\infty]{} (t^2 - 2t(n+1) + n(n+1))t^{(n-2)/2}e^{-t}.$$

Moreover, for all $d \in \mathbb{N}^*$,

$$\left| F_d(t) t^{(n-2)/2} \left(1 + \frac{t}{d}\right)^{-(n+1)/2} \right| \leqslant \left(1 + \frac{t}{d}\right)^{-d} t^{(n-2)/2}(4t^2 + (n+1)(3t+n)).$$

Let $d_0 > n/2 + 2$. Since $(1 + t/d)^{-d}$ is a non-increasing sequence of d, for all $d \geqslant d_0$,

$$\left| F_d(t) t^{(n-2)/2} \left(1 + \frac{t}{d}\right)^{-(n+1)/2} \right| \leqslant \left(1 + \frac{t}{d_0}\right)^{-d_0} t^{(n-2)/2}(4t^2 + (n+1)(3t+n)),$$

and the righthand side is integrable as a function of t. By Lebesgue's theorem, we have

$$\int_{t=0}^{+\infty} \frac{F_d(t) t^{(n-2)/2}}{(1 + t/d)^{(n+1)/2}} \,\mathrm{d}t \xrightarrow[d \to +\infty]{} \int_0^{+\infty} (t^2 - 2t(n+1) + n(n+1)) t^{(n-2)/2} e^{-t} \,\mathrm{d}t = \Gamma\left(\frac{n}{2} + 2\right), \tag{6.13}$$

where Γ is Euler's Gamma function. The conclusion follows from equations (6.8), (6.12), and (6.13). ◻

APPENDIX A. TECHNICAL COMPUTATIONS FOR SECTION 4

Before proving the technical lemmas of Section 4, we state several estimates that will be useful in this section and the next. Recall Definitions 4.4, 4.5, and 4.11. The following hold as t goes to infinity:

$$a(t) \xrightarrow[t \to +\infty]{} -1, \quad b_+(t) \xrightarrow[t \to +\infty]{} 0, \quad b_-(t) \xrightarrow[t \to +\infty]{} \sqrt{2}; \tag{A.1}$$

$$\begin{cases} u_i(t) \xrightarrow[t \to +\infty]{} 1 & \forall\, i \in \{1,2\}, \\ v_i(t) \xrightarrow[t \to +\infty]{} 1 & \forall\, i \in \{1,2,3,4\}. \end{cases} \tag{A.2}$$

The following hold as t goes to 0:

$$a(t) = 1 - \frac{t}{2} - \frac{t^2}{8} + O(t^3), \qquad (b_+(t))^2 = 2 - \frac{t}{2} + O(t^2), \tag{A.3}$$

$$b_+(t) b_-(t) = \sqrt{t}(1 + O(t^2)), \quad (b_-(t))^2 = \frac{t}{2} + \frac{t^2}{8} + O(t^3), \tag{A.4}$$

(A.5) $$u_1(t) = t + O(t^2), \qquad u_2(t) = \frac{t^2}{12} + O(t^3),$$

(A.6) $$v_1(t) = 2 + O(t), \qquad v_2(t) = \frac{t^3}{48} + O(t^4),$$

(A.7) $$v_3(t) = t + O(t^2), \qquad v_4(t) = 2 + O(t).$$

Proof of Lemma 4.6. Recall that P is defined by Definition 4.5 and $\tilde{\Omega}$ by equation (4.3). One can check by a direct computation that, for any $t \in [0, +\infty)$, $P(t) = (A(t) \otimes I_2)\sigma(Q \otimes I_2)$, where

$$A(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} b_-(t) & -b_+(t) \\ b_+(t) & b_-(t) \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Moreover, these three matrices are orthogonal. Then, we have

$$\begin{aligned}
&\sigma(Q \otimes I_2)\tilde{\Omega}(t)(Q^{\mathrm{t}} \otimes I_2)\sigma^{\mathrm{t}} \\
&= \sigma \begin{pmatrix} 1 - e^{-(1/2)t} & 0 & 0 & -\sqrt{t}e^{-(1/2)t} \\ 0 & 1 + e^{-(1/2)t} & \sqrt{t}e^{-(1/2)t} & 0 \\ 0 & \sqrt{t}e^{-(1/2)t} & 1 - (1-t)e^{-(1/2)t} & 0 \\ -\sqrt{t}e^{-(1/2)t} & 0 & 0 & 1 + (1-t)e^{-(1/2)t} \end{pmatrix} \sigma^{\mathrm{t}} \\
&= \begin{pmatrix} 1 - e^{-(1/2)t} & -\sqrt{t}e^{-(1/2)t} & 0 & 0 \\ -\sqrt{t}e^{-(1/2)t} & 1 + (1-t)e^{-(1/2)t} & 0 & 0 \\ 0 & 0 & 1 + e^{-(1/2)t} & \sqrt{t}e^{-(1/2)t} \\ 0 & 0 & \sqrt{t}e^{-(1/2)t} & 1 - (1-t)e^{-(1/2)t} \end{pmatrix} \\
&= I_4 + e^{-t/2}\left(\begin{pmatrix} 1 & \sqrt{t} \\ \sqrt{t} & t-1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right),
\end{aligned}$$

where I_4 stands for the identity matrix of size 4.

Recalling the definitions of $(v_i(t))_{1 \leqslant i \leqslant 4}$, $b_+(t)$, and $b_-(t)$ (see Definitions 4.4 and 4.5), we conclude the proof by checking that

$$A(t)\begin{pmatrix} 1 & \sqrt{t} \\ \sqrt{t} & t-1 \end{pmatrix}(A(t))^{\mathrm{t}} = \begin{pmatrix} \frac{t}{2} - \sqrt{1 + \left(\frac{t}{2}\right)^2} & 0 \\ 0 & \frac{t}{2} + \sqrt{1 + \left(\frac{t}{2}\right)^2} \end{pmatrix}. \qquad \square$$

Proof of Lemma 4.8. Let $z \in \mathbb{R}^n \setminus \{0\}$; by equation (4.2) and Lemma 4.3,

$$\det\left(\Omega(z)\right) = \det\left(\Omega'(z)\right)^r = \det\left(\tilde{\Omega}(\|z\|^2)\right) r(1 - e^{-\|z\|^2})^{r(n-1)},$$

and it is enough to prove that $\det(\tilde{\Omega}(t)) > 0$ whenever $t > 0$. By Lemma 4.6,

$$\text{(A.8)} \qquad \forall\, t \geqslant 0, \quad \det\left(\tilde{\Omega}(t)\right) = v_1(t)v_2(t)v_3(t)v_4(t) = 1 - (t^2+2)e^{-t} + e^{-2t} = f(t),$$

where the last equality defines $f : [0,+\infty) \to \mathbb{R}$. We have $f(0) = 0$ and, for all $t > 0$, $f'(t) = e^{-t}g(t)$ where $g(t) = t^2 - 2t + 2 - 2e^{-t}$. Then, $g(0) = 0$ and, for all $t > 0$, $g'(t) = 2(e^{-t} - 1 + t) > 0$. Thus, g is positive on $(0,+\infty)$, and so is f. Finally, we have that, for all $t > 0$, $\det(\tilde{\Omega}(t)) > 0$. □

Proof of Lemma 4.13. Let $z \in \mathbb{R}^n \setminus \{0\}$; as above, we have

$$\det\left(\Lambda(z)\right) = \det\left(\Lambda'(z)\right)^r = \det\left(\tilde{\Lambda}(\|z\|^2)\right)^r (1 - e^{-\|z\|^2})^{r(n-1)},$$

and it is enough to prove that $\det(\tilde{\Lambda}(t)) > 0$ whenever $t > 0$. By Lemma 4.12,

$$\forall\, t > 0, \quad \det\left(\tilde{\Lambda}(t)\right) = u_1(t)u_2(t) = \frac{1 - (t^2+2)e^{-t} + e^{-2t}}{1 - e^{-t}} = \frac{\det\left(\tilde{\Omega}(t)\right)}{1 - e^{-t}},$$

by equation (A.8). We just proved that $\det(\tilde{\Omega}(t))$ is positive for every positive t, hence the result. □

Proof of Lemma 4.14. First, recall that $\Omega(z) = \Omega'(z) \otimes I_r$ (see equation (4.2)) and $\Lambda(z) = \Lambda'(z) \otimes I_r$ (see equation (4.5)). Hence, we only need to prove that the map $z \longmapsto (0\ \Lambda'(z)^{1/2})\Omega'(z)^{-1/2}$ is bounded on $\mathbb{R}^n \setminus \{0\}$. Then, let $z \in \mathbb{R}^n \setminus \{0\}$; the matrix of $\Omega'(z)$ in the orthonormal basis $\mathcal{B}_z$ of $\mathbb{R}^2 \otimes (\mathbb{R} \oplus \mathbb{R}^n)$ (see Subsection 4.2) is given by Lemma 4.3, and the matrix of $\Omega'(z)^{-1/2}$ in $\mathcal{B}_z$ is

$$\left(\begin{array}{c|c} \tilde{\Omega}(\|z\|^2)^{-1/2} & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix}^{-1/2} \otimes I_{n-1} \end{array}\right).$$

Similarly, by Lemma 4.9, the matrix of $(\,0\ \Lambda'(z)^{1/2}\,)$ in $\mathcal{B}_z$ is

$$\left(\begin{array}{c|c} 0\ \tilde{\Lambda}(\|z\|^2)^{1/2} & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-(1/2)\|z\|^2} \\ e^{-(1/2)\|z\|^2} & 1 \end{pmatrix}^{1/2} \otimes I_{n-1} \end{array}\right).$$

Thus, our problem reduces to proving that $t \longmapsto \left(0\ \tilde{\Lambda}(t)^{1/2}\right)\tilde{\Omega}(t)^{-1/2}$ is bounded on $(0,+\infty)$.

Recall that, for all $t \in [0,+\infty)$, $P(t) \in O_4(\mathbb{R})$ was defined by Definition 4.5. By Lemmas 4.6 and 4.12, for all $t \in (0,+\infty)$ we have

$$\begin{aligned}
&\begin{pmatrix} 0 & \tilde{\Lambda}(t)^{1/2} \end{pmatrix} \tilde{\Omega}(t)^{-1/2} \\
&\quad = \left(0 \middle| Q^{\mathrm{t}} \begin{pmatrix} u_1(t)^{1/2} & 0 \\ 0 & u_2(t)^{1/2} \end{pmatrix} Q \right) P(t)^{\mathrm{t}} \\
&\qquad \times \begin{pmatrix} v_1(t)^{-1/2} & 0 & 0 & 0 \\ 0 & v_2(t)^{-1/2} & 0 & 0 \\ 0 & 0 & v_3(t)^{-1/2} & 0 \\ 0 & 0 & 0 & v_4(t)^{-1/2} \end{pmatrix} P(t) \\
&\quad = \begin{pmatrix} m_1(t) & m_3(t) & m_5(t) & m_6(t) \\ m_2(t) & m_4(t) & m_6(t) & m_5(t) \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
m_1 &= \frac{b_+ b_-}{4}\left(-\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_3}} + \sqrt{\frac{u_1}{v_4}}\right), \\
m_2 &= \frac{b_+ b_-}{4}\left(-\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_3}} - \sqrt{\frac{u_1}{v_4}}\right), \\
m_3 &= \frac{b_+ b_-}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_3}} + \sqrt{\frac{u_1}{v_4}}\right), \\
m_4 &= \frac{b_+ b_-}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_3}} - \sqrt{\frac{u_1}{v_4}}\right), \\
m_5 &= \frac{(b_+)^2}{4}\left(\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_1}{v_3}}\right) + \frac{(b_-)^2}{4}\left(\sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_4}}\right), \\
m_6 &= \frac{(b_+)^2}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_1}{v_3}}\right) + \frac{(b_-)^2}{4}\left(\sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_4}}\right).
\end{aligned}$$

By Lemma 4.8, for all $t > 0$, the $(v_i(t))_{1 \leqslant i \leqslant 4}$ are the eigenvalues of a symmetric positive operator, and hence are positive. Similarly, for all $t > 0$, $u_1(t) > 0$, and $u_2(t) > 0$ by Lemma 4.13. Thus, the $(m_i)_{1 \leqslant i \leqslant 6}$ are well-defined continuous maps from $(0,+\infty)$ to $\mathbb{R}$. By equations (A.1) and (A.2),

$$m_i(t) \xrightarrow[t \to +\infty]{} \begin{cases} 0 & \forall\, i \in \{1,2,3,4,6\}, \\ 1 & \text{for } i = 5. \end{cases}$$

Moreover, by equations (A.3)–(A.7), for all $i \in \{1,2,5,6\}$, $m_i(t) = \frac{1}{2} + O(\sqrt{t})$ as t goes to 0, and, for any $i \in \{3,4\}$, $m_i(t) = -\frac{1}{2} + O(\sqrt{t})$ as t goes to 0.

Hence, for all $i \in \{1, \dots, 6\}$, m_i is a bounded function from $(0, +\infty)$ to $\mathbb{R}$, which concludes the proof. ◻

Proof of Lemma 4.17. Recall that, for all $t > 0$, the couples $(X_{ij}(t), Y_{ij}(t))$ are independent centered Gaussian vectors in $\mathbb{R}^2$. We denote by $\Lambda_{ij}(t)$ the variance matrix of $(X_{ij}(t), Y_{ij}(t))$, which equals $\tilde{\Lambda}(t)$ if $j = 1$, and

$$\begin{pmatrix} 1 & \exp^{-(1/2)t^2} \\ \exp^{-(1/2)t^2} & 1 \end{pmatrix}$$

otherwise (see Definition 1.5, Lemma 4.16, and Lemma 4.9).

For all $i \in \{1, \dots, r\}$, $j \in \{1, \dots, n\}$, and $t > 0$, we can write

$$\begin{pmatrix} X_{ij}(t) \\ Y_{ij}(t) \end{pmatrix} = \sqrt{\Lambda_{ij}(t)} \begin{pmatrix} A_{ij} \\ B_{ij} \end{pmatrix},$$

where the (A_{ij}) and (B_{ij}) are globally independent real standard Gaussian variables, not depending on t. Note that by Lemma 4.13, the $\Lambda_{ij}(t)$ are positive for any $t > 0$. We deduce from Lemma 4.12 that, for any $i \in \{1, \dots, r\}$, for all $t > 0$,

$$\sqrt{\Lambda_{i1}(t)} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \alpha(t) \end{pmatrix},$$

$$\sqrt{\Lambda_{ij}(t)} = \begin{pmatrix} \gamma(t) & \delta(t) \\ \delta(t) & \gamma(t) \end{pmatrix}, \quad \forall\, j \in \{2, \dots, n\},$$

where

$$\text{(A.9)} \qquad \begin{aligned} \alpha(t) &= \frac{1}{2}\left(\sqrt{u_2(t)} + \sqrt{u_1(t)}\right), \\ \gamma(t) &= \frac{1}{2}\left(\sqrt{1 + e^{-(1/2)t^2}} + \sqrt{1 - e^{-(1/2)t^2}}\right), \end{aligned}$$

$$\text{(A.10)} \qquad \begin{aligned} \beta(t) &= \frac{1}{2}\left(\sqrt{u_2(t)} - \sqrt{u_1(t)}\right), \\ \delta(t) &= \frac{1}{2}\left(\sqrt{1 + e^{-(1/2)t^2}} - \sqrt{1 - e^{-(1/2)t^2}}\right). \end{aligned}$$

We denote $A_j = (A_{1j}, \dots, A_{rj})^{\mathrm{t}}$ the j-th column of A, and similarly $B_j = (B_{1j}, \dots, B_{rj})^{\mathrm{t}}$. Then, $\mathbb{E}[|\det^{\perp}(X(t))||\det^{\perp}(Y(t))|] = \mathbb{E}[\Psi(t, A, B)]$, where

$$\text{(A.11)} \quad \begin{aligned} &\Psi(t, A, B) \\ &= |\det^{\perp}(\alpha(t)A_1 + \beta(t)B_1, \gamma(t)A_2 + \delta(t)B_2, \dots, \gamma(t)A_n + \delta(t)B_n)| \\ &\quad \times |\det^{\perp}(\beta(t)A_1 + \alpha(t)B_1, \delta(t)A_2 + \gamma(t)B_2, \dots, \delta(t)A_n + \gamma(t)B_n)|. \end{aligned}$$

By (A.5), $\alpha(t) = \frac{1}{2}\sqrt{t} + O(t)$ and $\beta(t) = -\frac{1}{2}\sqrt{t} + O(t)$. We extend continuously α, β, γ, and δ by $\alpha(0) = 0 = \beta(0)$ and $\gamma(0) = 1/\sqrt{2} = \delta(0)$. The function Ψ also extends continuously at $t = 0$.

Then, α, β, γ, and δ are bounded functions on $(0,1]$, and Ψ is the square root of a polynomial of degree $4r$ in (A, B) whose coefficients are bounded functions of t. In particular, for all $t \in (0,1]$, $\Psi(t, A, B)$ is dominated by a polynomial in (A, B) whose coefficients are independent of t. By Lebesgue's theorem,

$$\mathbb{E}[|\det{}^{\perp}(X(t))|\,|\det{}^{\perp}(Y(t))|] \xrightarrow[t\to 0]{} \mathbb{E}[\Psi(0, A, B)]. \tag{A.12}$$

Let $j \in \{2, \dots, n\}$; we define $X_j = (X_{1j}, \dots, X_{rj})^{\mathrm{t}}$ as

$$\gamma(0)A_j + \delta(0)B_j = \frac{1}{\sqrt{2}}(A_j + B_j).$$

Then, the (X_{ij}) with $i \in \{1, \dots, r\}$ and $j \in \{2, \dots, n\}$ are independent real standard Gaussian variables. Setting $X_1 = (X_{11}, \dots, X_{r1})^{\mathrm{t}} = 0$, we have

$$\begin{aligned}\Psi(0, A, B) &= |\det{}^{\perp}(X_1, X_2, \dots, X_n)|^2 \\ &= \det((X_1, X_2, \dots, X_n)(X_1, X_2, \dots, X_n)^{\mathrm{t}}) \\ &= \sum_{1 \leqslant k_1 < \cdots < k_r \leqslant n} \det((X_{ik_j})_{1 \leqslant i,j \leqslant r})^2,\end{aligned}$$

by the Cauchy-Binet formula. Let $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n$. If $k_1 = 1$, the first column of $(X_{ik_j})_{1 \leqslant i,j \leqslant r}$ is zero and its determinant equals 0. Otherwise,

$$\mathbb{E}[\det((X_{ik_j})_{1 \leqslant i,j \leqslant r})^2] = \sum_{\sigma,\tau \in \mathfrak{S}_r} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^{r} \mathbb{E}[X_{ik_{\sigma(i)}} X_{ik_{\tau(i)}}] = r!. \tag{A.13}$$

Hence, if $r < n$,

$$\mathbb{E}[\Psi(0, A, B)] = r!\binom{n-1}{r} = \frac{(n-1)!}{(n-r-1)!},$$

and by equation (A.12), we have proven Lemma 4.17 in this case. If $r = n$, we have $\mathbb{E}[\Psi(0, A, B)] = 0$ and must be more precise.

Let us now assume that $r = n$. Then, X and Y are square matrices and their Jacobians are simply the absolute values of their determinants. For all $t > 0$,

$$\begin{aligned}&\Psi(t, A, B) \\ &= \frac{t}{2}\left|\det\left(\sqrt{\frac{2}{t}}\alpha(t)A_1 + \sqrt{\frac{2}{t}}\beta(t)B_1, \gamma(t)A_2 + \delta(t)B_2, \dots, \gamma(t)A_n + \delta(t)B_n\right)\right| \\ &\quad \times \left|\det\left(\sqrt{\frac{2}{t}}\beta(t)A_1 + \sqrt{\frac{2}{t}}\alpha(t)B_1, \delta(t)A_2 + \gamma(t)B_2, \dots, \delta(t)A_n + \gamma(t)B_n\right)\right|.\end{aligned} \tag{A.14}$$

By equation (A.5), $\sqrt{2/t}\alpha(t) = 1/\sqrt{2}+O(\sqrt{t})$ and $\sqrt{2/t}\beta(t) = -1/\sqrt{2}+O(\sqrt{t})$. We can apply the same kind of argument as above. By Lebesgue's theorem,

$$\begin{aligned}\frac{2}{t}\mathbb{E}[\Psi(t,A,B)] \xrightarrow[t\to 0]{} & \mathbb{E}[|\det(Y_1,X_2,\dots,X_n)|\,|\det(-Y_1,X_2,\dots,X_n)|] \\ & = \mathbb{E}[\det(Y_1,X_2,\dots,X_n)^2],\end{aligned}$$

where $Y_1 = (Y_{11},\dots,Y_{r1})^{\mathrm{t}} = (1/\sqrt{2})(A_1 - B_1)$. Since $Y_1, X_2, \dots, X_n$ are independent $\mathcal{N}(\mathrm{Id})$ in $\mathbb{R}^r$, the same computation as equation (A.13) shows that $\mathbb{E}[\det(Y_1,X_2,\dots,X_n)^2] = r! = n!$. Hence, if $r = n$, we have

$$\mathbb{E}[|\det{}^{\perp}(X(t))|\,|\det{}^{\perp}(Y(t))|] = \mathbb{E}[\Psi(t,A,B)] \sim \frac{n!}{2}t, \quad \text{as } t \to 0. \qquad \square$$

Proof of Lemma 4.18. For any $t > 0$, let us denote by

$$\text{(A.15)} \qquad \hat{\Lambda}(t) = \left(\begin{array}{c|c} \tilde{\Lambda}(t) & 0 \\ \hline 0 & \begin{pmatrix} 1 & \exp^{-(1/2)t^2} \\ \exp^{-(1/2)t^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right) \otimes I_r$$

the variance matrix of $(X(t), Y(t))$.

In the following, we denote a generic element of $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ by $L = (X, Y)$. We have

$$\text{(A.16)} \quad \begin{aligned}\mathbb{E}[|\det{}^{\perp}(X(t))|\,|\det{}^{\perp}(Y(t))|] &= \frac{1}{(2\pi)^{rn}} \det\left(\hat{\Lambda}(t)\right)^{-1/2} \\ &\times \int |\det{}^{\perp}(X)|\,|\det{}^{\perp}(Y)| \exp\left(-\frac{1}{2}\left\langle \left(\hat{\Lambda}(t)\right)^{-1} L, L\right\rangle\right) \mathrm{d}L.\end{aligned}$$

By Lemma 4.12, we have $\hat{\Lambda}(t) = \mathrm{Id} + O(te^{-t/2})$ as $t \to +\infty$. Then, we have $\det(\hat{\Lambda}(t))^{-1/2} = 1 + O(te^{-t/2})$. Moreover, by the Mean Value Theorem,

$$\begin{aligned}&\left|\exp\left(-\frac{1}{2}\left\langle\left(\hat{\Lambda}(t)\right)^{-1} L, L\right\rangle\right) - e^{-(1/2)\|L\|^2}\right| \\ &\quad = e^{-(1/2)\|L\|^2}\left|\exp\left(-\frac{1}{2}\left\langle\left(\left(\hat{\Lambda}(t)\right)^{-1} - \mathrm{Id}\right) L, L\right\rangle\right) - 1\right| \\ &\quad \leqslant e^{-(1/2)\|L\|^2}\frac{\|L\|^2}{2}\left\|\left(\hat{\Lambda}(t)\right)^{-1} - \mathrm{Id}\right\| \exp\left(\frac{\|L\|^2}{2}\left\|\left(\hat{\Lambda}(t)\right)^{-1} - \mathrm{Id}\right\|\right).\end{aligned}$$

Then, since $(\hat{\Lambda}(t))^{-1} = \mathrm{Id} + O(te^{-t/2})$, this last term is smaller than

$$e^{-(1/4)\|L\|^2}\frac{\|L\|^2}{2}\left\|\left(\hat{\Lambda}(t)\right)^{-1} - \mathrm{Id}\right\|$$

for all t large enough. Hence,

$$\begin{aligned}\int |\det{}^\perp(X)|\,|\det{}^\perp(Y)| \left| \exp\left(-\frac{1}{2}\left\langle \left(\hat{\Lambda}(t)\right)^{-1} L, L\right\rangle\right) - e^{-(1/2)\|L\|^2} \right| \mathrm{d}L \\ \leqslant \frac{1}{2}\left\|\left(\hat{\Lambda}(t)\right)^{-1} - \mathrm{Id}\right\| \int |\det{}^\perp(X)|\,|\det{}^\perp(Y)|\, \|L\|^2 e^{-(1/4)\|L\|^2}\,\mathrm{d}L \\ = O(te^{-t/2}).\end{aligned}$$

Thanks to this relation and equation (A.16), we get that

$$\begin{aligned}&\mathbb{E}[|\det{}^\perp(X(t))|\,|\det{}^\perp(Y(t))|] \\ &\qquad = \mathbb{E}[|\det{}^\perp(X(\infty))|\,|\det{}^\perp(Y(\infty))|] + O(te^{-t/2}),\end{aligned}$$

where $(X(\infty), Y(\infty)) \sim \mathcal{N}(\mathrm{Id})$ in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$. Finally, by Lemma A.14 of [21],

$$\begin{aligned}&\mathbb{E}[|\det{}^\perp(X(\infty))|\,|\det{}^\perp(Y(\infty))|] \\ &\qquad = \mathbb{E}[|\det{}^\perp(X(\infty))|]^2 = (2\pi)^r \left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2.\end{aligned}$$

□

Appendix B. Technical Computations for Section 5

Proof of Lemma 5.26. Let $\alpha \in (0,1)$; we want to prove that

$$\Theta(z)^{-1/2}\Theta_d(z)\Theta(z)^{-1/2} - \mathrm{Id} = O(d^{-\alpha})$$

uniformly for $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$. Recall that

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}.$$

Since $Q \in O_2(\mathbb{R})$, it is equivalent to prove that

$$\begin{aligned}(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x})\Theta(z)^{-1/2}(\Theta_d(z) - \Theta(z)) \\ \times \Theta(z)^{-1/2}(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x})^{-1} = O(d^{-\alpha}).\end{aligned} \tag{B.1}$$

Recall that e_d was defined by equation (3.1) and $e_\infty = \xi\,\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$ (see Section 4). We set $\varepsilon_d(w,z) = e_d(w,z) - e_\infty(w,z)$, for any $d \in \mathbb{N}$, for all $x \in M$, and for all $w, z \in B_{T_xM}(0, b_n \ln d)$. By equations (4.1) and (5.7) we have

$$\Theta_d(z) - \Theta(z) = \begin{pmatrix} \varepsilon_d(0,0) & \varepsilon_d(0,z) \\ \varepsilon_d(z,0) & \varepsilon_d(z,z)\end{pmatrix}.$$

Then, by Lemma 4.1, for all $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have

$$\begin{aligned}(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x})\Theta(z)^{-1/2}(\Theta_d(z) - \Theta(z))\\ \times \Theta(z)^{-1/2}(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x})^{-1} = \begin{pmatrix} a_d(z) & b_d(z)^* \\ b_d(z) & c_d(z)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned} a_d(z) &= \frac{1}{2}(1 - e^{-(1/2)\|z\|^2})^{-1}(\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)),\\ b_d(z) &= -\frac{1}{2}(1 - e^{-\|z\|^2})^{-1/2}(\varepsilon_d(z,z) - \varepsilon_d(z,0) + \varepsilon_d(0,z) - \varepsilon_d(0,0)),\\ c_d(z) &= \frac{1}{2}(1 + e^{-(1/2)\|z\|^2})^{-1}(\varepsilon_d(z,z) + \varepsilon_d(z,0) + \varepsilon_d(0,z) + \varepsilon_d(0,0)).\end{aligned}$$

Let $\beta \in (\alpha, 1)$; by Proposition 3.4 we have $\|\mathrm{D}^2_{(w,z)}\varepsilon_d\| \leqslant Cd^{-\beta}$, where C is independent of $x \in M$ and $w, z \in B_{T_xM}(0, b_n \ln d)$. Then, a second-order Taylor expansion around $(0,0)$ gives

$$\|\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)\| \leqslant C\|z\|^2 d^{-\beta}.$$

Since we consider $z \in B_{T_xM}(0, b_n \ln d)$ and $1 - e^{-(1/2)\|z\|^2} \sim \|z\|^2/2$ as $z \to 0$, we have

$$\|a_d(z)\| \leqslant \frac{C\|z\|^2 d^{-\beta}}{2(1 - e^{-(1/2)\|z\|^2})} = O((\ln d)^2 d^{-\beta}) = O(d^{-\alpha}),$$

where the error term does not depend on (x, z). We obtain equation (B.1) by reasoning similarly for $b_d(z)$ and $c_d(z)$. ◻

Proof of Lemma 5.28. The idea of the proof is the same as that of Lemma 5.26 above. Let $\alpha \in (0,1)$; we want to prove that

$$\Omega(z)^{-1/2}(\Omega_d(z) - \Omega(z))\Omega(z)^{-1/2} = O(d^{-\alpha}). \tag{B.2}$$

Recall that we defined $\varepsilon_d(w,z) = e_d(w,z) - e_\infty(w,z)$ for any $x \in M$ and $w, z \in B_{T_xM}(0, b_n \ln d)$. We can express $\Omega_d(z) - \Omega(z)$ in terms of ε_d and its derivatives. Then, we write the matrix of the lefthand side of equation (B.2) in an orthonormal basis that diagonalizes $\Omega(z)$. The coefficients of this matrix are linear combinations of ε_d and its derivatives. We will prove that they are $O(d^{-\alpha})$ using Taylor expansions and the estimates of Subsection 3.3.

The details are longer than in the proof of Lemma 5.26 for two reasons. First, the basis in which $\Omega(z)$ is diagonal now depends on z. Second, some of the eigenvalues of $\Omega(z)$ are $O(\|z\|^6)$ as $z \to 0$, so that we need to consider Taylor expansions of order 6 for some coefficients. In addition, the matrices involved are less easily described than in the proof of Lemma 5.26.

Recall that e_d was defined by equation (3.1) and that $e_\infty = \xi \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$ (see Section 4). We expressed $\Omega(z)$ in terms of e_∞ in equation (4.2) and $\Omega_d(z)$ in terms of e_d in equation (5.8). As an operator on

$$\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \oplus \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \oplus (T_x^*M \otimes \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x) \oplus (T_x^*M \otimes \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x),$$

we have

$$\Omega_d(z) - \Omega(z) = \left(\begin{array}{cc|cc} \varepsilon_d(0,0) & \varepsilon_d(0,z) & \partial_y^\sharp \varepsilon_d(0,0) & \partial_y^\sharp \varepsilon_d(0,z) \\ \varepsilon_d(z,0) & \varepsilon_d(z,z) & \partial_y^\sharp \varepsilon_d(z,0) & \partial_y^\sharp \varepsilon_d(z,z) \\ \hline \partial_x \varepsilon_d(0,0) & \partial_x \varepsilon_d(0,z) & \partial_x \partial_y^\sharp \varepsilon_d(0,0) & \partial_x \partial_y^\sharp \varepsilon_d(0,z) \\ \partial_x \varepsilon_d(z,0) & \partial_x \varepsilon_d(z,z) & \partial_x \partial_y^\sharp \varepsilon_d(z,0) & \partial_x \partial_y^\sharp \varepsilon_d(z,z) \end{array}\right).$$

Let us choose an orthonormal basis $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ of T_xM such that $z = \|z\|\, \partial/\partial x_1$. We denote by $(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ its dual basis. We can then define a basis of $\mathbb{R}^2 \otimes (\mathbb{R} \oplus T_x^*M)$ similar to $\mathcal{B}_z$ (see Subsection 4.2). For any $i \in \{1, \dots, n\}$, we denote by ∂_{x_i} (respectively, ∂_{y_i}) the partial derivative with respect to the i-th component of the first (respectively, second) variable for maps from $T_xM \times T_xM$ to $\operatorname{End}(\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x)$. Then, we can split $\Omega_d(z) - \Omega(z)$ according to the previous basis in the following way:

$$\Omega_d(z) - \Omega(z) = \begin{pmatrix} A_d(z) & B_d^{(1)}(z)^* & \cdots & B_d^{(n)}(z)^* \\ B_d^{(1)}(z) & C_d^{(11)}(z) & \cdots & C_d^{(1n)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ B_d^{(n)}(z) & C_d^{(n1)}(z) & \cdots & C_d^{(nn)}(z) \end{pmatrix}, \tag{B.3}$$

where

$$A_d(z) = \begin{pmatrix} \varepsilon_d(0,0) & \varepsilon_d(0,z) \\ \varepsilon_d(z,0) & \varepsilon_d(z,z) \end{pmatrix}, \tag{B.4}$$

$$B_d^{(i)}(z) = \begin{pmatrix} \partial_{x_i}\varepsilon_d(0,0) & \partial_{x_i}\varepsilon_d(0,z) \\ \partial_{x_i}\varepsilon_d(z,0) & \partial_{x_i}\varepsilon_d(z,z) \end{pmatrix}, \qquad \forall\, i \in \{1, \dots, n\}, \tag{B.5}$$

$$C_d^{(ij)}(z) = \begin{pmatrix} \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(0,0) & \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(0,z) \\ \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(z,0) & \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(z,z) \end{pmatrix}, \quad \forall\, i,j \in \{1, \dots, n\},. \tag{B.6}$$

Let us denote by $\mathcal{P}(z)$ the operator whose matrix in our basis is

$$\left(\begin{array}{c|c} P(\|z\|^2) & 0 \\ \hline 0 & Q \otimes I_{n-1} \end{array}\right) \otimes I_r,$$

where P was defined by Definition 4.5 and $Q = (1/\sqrt{2})\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Since $\mathcal{P}(z)$ is orthogonal, (B.2) is equivalent to the following:

$$\mathcal{P}(z)\Omega(z)^{-1/2}(\Omega_d(z) - \Omega(z))\Omega(z)^{-1/2}\mathcal{P}(z)^{-1} = O(d^{-\alpha}). \tag{B.7}$$

By Lemma 4.7, the matrix of $\mathcal{P}(z)\Omega(z)^{-1/2}\mathcal{P}(z)^{-1}$ is $\left(\begin{smallmatrix} V(z) & 0 \\ 0 & N(z)\otimes I_{n-1} \end{smallmatrix}\right)$, where

$$V(z) = \begin{pmatrix} v_1(\|z\|^2)^{-1/2} & 0 & 0 & 0 \\ 0 & v_2(\|z\|^2)^{-1/2} & 0 & 0 \\ 0 & 0 & v_3(\|z\|^2)^{-1/2} & 0 \\ 0 & 0 & 0 & v_4(\|z\|^2)^{-1/2} \end{pmatrix} \otimes I_r,$$

$$N(z) = \begin{pmatrix} (1 - e^{-(1/2)\|z\|^2})^{-1/2} & 0 \\ 0 & (1 + e^{-(1/2)\|z\|^2})^{-1/2} \end{pmatrix} \otimes I_r.$$

On the other hand, by equation (B.3),

$$\mathcal{P}(z)(\Omega_d(z) - \Omega(z))\mathcal{P}(z)^{-1} = \begin{pmatrix} \tilde{A}_d(z) & \tilde{B}_d^{(1)}(z)^* & \cdots & \tilde{B}_d^{(n)}(z)^* \\ \tilde{B}_d^{(1)}(z) & \tilde{C}_d^{(11)}(z) & \cdots & \tilde{C}_d^{(1n)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_d^{(n)}(z) & \tilde{C}_d^{(n1)}(z) & \cdots & \tilde{C}_d^{(nn)}(z) \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{A}_d(z) & \tilde{B}_d^{(1)}(z)^* \\ \tilde{B}_d^{(1)}(z) & \tilde{C}_d^{(11)}(z) \end{pmatrix} = (P(\|z\|^2) \otimes \mathrm{Id}) \begin{pmatrix} A_d(z) & B_d^{(1)}(z)^* \\ B_d^{(1)}(z) & C_d^{(11)}(z) \end{pmatrix} \times (P(\|z\|^2)^{\mathrm{t}} \otimes \mathrm{Id}), \tag{B.8}$$

$$\left(\tilde{B}_d^{(i)}(z)\ \tilde{C}_d^{(i1)}(z)\right) = (Q \otimes \mathrm{Id}) \left(B_d^{(i)}(z)\ C_d^{(i1)}(z)\right) \times (P(\|z\|^2)^{\mathrm{t}} \otimes \mathrm{Id}), \quad \forall\, i \in \{2, \dots, n\}, \tag{B.9}$$

$$\tilde{C}_d^{(ij)}(z) = (Q \otimes \mathrm{Id})C_d^{(ij)}(z)(Q^{\mathrm{t}} \otimes \mathrm{Id}), \quad \forall\, i, j \in \{2, \dots, n\}. \tag{B.10}$$

Then, in order to prove equation (B.7), we have to prove that

$$V(z) \begin{pmatrix} \tilde{A}_d(z) & \tilde{B}_d^{(1)}(z)^* \\ \tilde{B}_d^{(1)}(z) & \tilde{C}_d^{(11)}(z) \end{pmatrix} V(z) = O(d^{-\alpha}), \tag{B.11}$$

$$N(z) \left(\tilde{B}_d^{(i)}(z)\ \tilde{C}_d^{(i1)}(z)\right) V(z) = O(d^{-\alpha}), \qquad \forall\, i \in \{2, \dots, n\}, \tag{B.12}$$

$$N(z)\tilde{C}_d^{(ij)}(z)N(z) = O(d^{-\alpha}), \qquad \forall\, i, j \in \{2, \dots, n\}. \tag{B.13}$$

Since these are heavy computations, we do not reproduce them in totality here. In the following, we give some details about the proof of (B.11), which is the most difficult of these three relations to establish. The proofs of (B.12) and (B.13) are similar and left to the fearless reader.

Let us focus on the proof of (B.11). We denote

$$V(z)\begin{pmatrix} \tilde{A}_d(z) & \tilde{B}_d^{(1)}(z)^* \\ \tilde{B}_d^{(1)}(z) & \tilde{C}_d^{(11)}(z) \end{pmatrix} V(z) = \begin{pmatrix} a_d^{(1)} & a_d^{(2)*} & b_d^{(1)*} & b_d^{(2)*} \\ a_d^{(2)} & a_d^{(3)} & b_d^{(3)*} & b_d^{(4)*} \\ b_d^{(1)} & b_d^{(3)} & c_d^{(1)} & c_d^{(2)*} \\ b_d^{(2)} & b_d^{(4)} & c_d^{(2)} & c_d^{(3)} \end{pmatrix}.$$

Then, by Definition 4.5 and equations (B.4), (B.5), (B.6), and (B.8), we have

$$\begin{aligned}
\text{(B.14)}\quad a_d^{(1)}(z) = \frac{1}{4}(v_1(\|z\|^2))^{-1}\Big(& (b_+(\|z\|^2))^2 \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z) - \partial_{x_1}\varepsilon_d(z,0) + \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) + \partial_{y_1}^\sharp\varepsilon_d(z,0) - \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ (b_-(\|z\|^2))^2\left(\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\text{(B.15)}\quad a_d^{(2)}(z) = \frac{1}{4}(v_1(\|z\|^2)v_2(\|z\|^2))^{-1/2}\Big(& -b_+(\|z\|^2)b_-(\|z\|^2) \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&- (b_-(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z) - \partial_{x_1}\varepsilon_d(z,0) + \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&+ (b_+(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) + \partial_{y_1}^\sharp\varepsilon_d(z,0) - \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\text{(B.16)}\quad a_d^{(3)}(z) = \frac{1}{4}(v_2(\|z\|^2))^{-1}\Big(& (b_-(\|z\|^2))^2 \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&- b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z) - \partial_{x_1}\varepsilon_d(z,0) + \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&- b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) + \partial_{y_1}^\sharp\varepsilon_d(z,0) - \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ (b_+(\|z\|^2))^2\left(\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big),
\end{aligned}$$

$$\begin{aligned}\text{(B.17)}\quad b_d^{(1)}(z) &= \frac{1}{4}(v_1(\|z\|^2)v_3(\|z\|^2))^{-1/2}\Big(-(b_+(\|z\|^2))^2 \\ &\times \Big(\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,0)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,z)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- b_+(\|z\|^2)b_-(\|z\|^2)\Big(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\Big) \\ &- b_+(\|z\|^2)b_-(\|z\|^2)\Big(\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial^\sharp_{y_1}\varepsilon_d(z,0)+\partial^\sharp_{y_1}\varepsilon_d(0,z)+\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- (b_-(\|z\|^2))^2\Big(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\Big)\Big),\end{aligned}$$

$$\begin{aligned}\text{(B.18)}\quad b_d^{(2)}(z) &= \frac{1}{4}(v_1(\|z\|^2)v_4(\|z\|^2))^{-1/2}\Big(b_+(\|z\|^2)b_-(\|z\|^2) \\ &\times \Big(\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,0)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,z)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &+ (b_-(\|z\|^2))^2\Big(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\Big) \\ &- (b_+(\|z\|^2))^2\Big(\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial^\sharp_{y_1}\varepsilon_d(z,0)+\partial^\sharp_{y_1}\varepsilon_d(0,z)+\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- b_+(\|z\|^2)b_-(\|z\|^2)\Big(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\Big)\Big),\end{aligned}$$

$$\begin{aligned}\text{(B.19)}\quad b_d^{(3)}(z) &= \frac{1}{4}(v_2(\|z\|^2)v_3(\|z\|^2))^{-1/2}\Big(b_+(\|z\|^2)b_-(\|z\|^2) \\ &\times \Big(\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,0)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,z)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- (b_+(\|z\|^2))^2\Big(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\Big) \\ &+ (b_-(\|z\|^2))^2\Big(\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial^\sharp_{y_1}\varepsilon_d(z,0)+\partial^\sharp_{y_1}\varepsilon_d(0,z)+\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- b_+(\|z\|^2)b_-(\|z\|^2)\Big(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\Big)\Big),\end{aligned}$$

$$\begin{aligned}\text{(B.20)}\quad b_d^{(4)}(z) &= \frac{1}{4}(v_2(\|z\|^2)v_4(\|z\|^2))^{-1/2}\Big(-(b_-(\|z\|^2))^2 \\ &\times \Big(\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,0)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,z)-\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &+ b_+(\|z\|^2)b_-(\|z\|^2)\Big(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\Big) \\ &+ b_+(\|z\|^2)b_-(\|z\|^2)\Big(\partial^\sharp_{y_1}\varepsilon_d(z,z)+\partial^\sharp_{y_1}\varepsilon_d(z,0)+\partial^\sharp_{y_1}\varepsilon_d(0,z)+\partial^\sharp_{y_1}\varepsilon_d(0,0)\Big) \\ &- (b_+(\|z\|^2))^2\Big(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\Big)\Big),\end{aligned}$$

$$\begin{aligned}
\text{(B.21)}\quad c_d^{(1)}(z) &= \frac{1}{4}(v_3(\|z\|^2))^{-1}\Big((b_+(\|z\|^2))^2 \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z) + \partial_{x_1}\varepsilon_d(z,0) - \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) - \partial_{y_1}^\sharp\varepsilon_d(z,0) + \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ (b_-(\|z\|^2))^2\left(\varepsilon_d(z,z) + \varepsilon_d(z,0) + \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\text{(B.22)}\quad c_d^{(2)}(z) &= \frac{1}{4}(v_3(\|z\|^2)v_4(\|z\|^2))^{-1/2}\Big(-b_+(\|z\|^2)b_-(\|z\|^2) \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&- (b_-(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z) + \partial_{x_1}\varepsilon_d(z,0) - \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&+ (b_+(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) - \partial_{y_1}^\sharp\varepsilon_d(z,0) + \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z) + \varepsilon_d(z,0) + \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\text{(B.23)}\quad c_d^{(3)}(z) &= \frac{1}{4}(v_4(\|z\|^2))^{-1}\Big((b_-(\|z\|^2))^2 \\
&\times \left(\partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,z) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(z,0) - \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,z) + \partial_{x_1}\partial_{y_1}^\sharp \varepsilon_d(0,0)\right) \\
&- b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z) + \partial_{x_1}\varepsilon_d(z,0) - \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0)\right) \\
&- b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z) - \partial_{y_1}^\sharp\varepsilon_d(z,0) + \partial_{y_1}^\sharp\varepsilon_d(0,z) - \partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\
&+ (b_+(\|z\|^2))^2\left(\varepsilon_d(z,z) + \varepsilon_d(z,0) + \varepsilon_d(0,z) + \varepsilon_d(0,0)\right)\Big).
\end{aligned}$$

We need to prove that each one of the terms (B.14) to (B.23) is a $O(d^{-\alpha})$, where the constant involved in this notation is independent of (x, z). The main difficulty comes from the fact that v_2 and v_3 converge to 0 as $z \to 0$ (see equations (A.6) and (A.7)).

The term with the worst apparent singularity at $z = 0$ is $a_d^{(3)}$ (see (B.16)). We will show below that $a_d^{(3)}(z) = O(d^{-\alpha})$ uniformly in (x, z). The proofs that the other nine coefficients are $O(d^{-\alpha})$ follow the same lines, and they are strictly easier technically. We leave them to the reader.

By equation (A.6), $v_2(\|z\|^2) \sim \|z\|^6/48$ as $z \to 0$. Hence, we have to expand the second factor in (B.16) up to a $O(\|z\|^6)$. Let $\beta \in (\alpha, 1)$, and recall that, by Proposition 3.4, the partial derivatives of ε_d of order up to 6 are $O(d^{-\beta})$ uniformly on $B_{T_xM}(0, b_n \ln d) \times B_{T_xM}(0, b_n \ln d)$. Recall also that we chose our coordinates so that $z = (\|z\|, 0, \dots, 0)$. Using Taylor expansions around $(0,0)$ for ε_d and its derivatives, we get

$$\begin{aligned}
\text{(B.24)}\quad &\partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(z,z) \\
&\quad + \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(z,0) + \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,z) + \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) \\
&= 4\partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + 2\|z\| \left(\partial^2_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \partial_{x_1} \left(\partial^\sharp_{y_1}\right)^2 \varepsilon_d(0,0)\right) \\
&\quad + \|z\|^2 (\partial^3_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \partial^2_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) + \partial_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0)) \\
&\quad + \|z\|^3 \Big(\frac{1}{3} \partial^4_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{2} \partial^3_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) \\
&\quad + \frac{1}{2} \partial^2_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0) + \frac{1}{3} \partial_{x_1} (\partial^\sharp_{y_1})^4 \varepsilon_d(0,0)\Big) + \|z\|^4 O(d^{-\beta}),
\end{aligned}$$

$$\begin{aligned}
\text{(B.25)}\quad &\partial_{x_1} \varepsilon_d(z,z) - \partial_{x_1} \varepsilon_d(z,0) + \partial_{x_1} \varepsilon_d(0,z) - \partial_{x_1} \varepsilon_d(0,0) \\
&= 2\|z\| \, \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \|z\|^2 (\partial^2_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \partial_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0)) \\
&\quad + \|z\|^3 \left(\frac{1}{2} \partial^3_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{2} \partial^2_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) + \frac{1}{3} \partial_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0)\right) \\
&\quad + \|z\|^4 \Big(\frac{1}{6} \partial^4_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{4} \partial^3_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) \\
&\quad + \frac{1}{6} \partial^2_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0) + \frac{1}{12} \partial_{x_1} (\partial^\sharp_{y_1})^4 \varepsilon_d(0,0)\Big) + \|z\|^5 O(d^{-\beta}),
\end{aligned}$$

$$\begin{aligned}
\text{(B.26)}\quad &\partial^\sharp_{y_1} \varepsilon_d(z,z) + \partial^\sharp_{y_1} \varepsilon_d(z,0) - \partial^\sharp_{y_1} \varepsilon_d(0,z) - \partial^\sharp_{y_1} \varepsilon_d(0,0) \\
&= 2\|z\| \, \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \|z\|^2 (\partial^2_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \partial_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0)) \\
&\quad + \|z\|^3 \left(\frac{1}{3} \partial^3_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{2} \partial^2_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) + \frac{1}{2} \partial_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0)\right) \\
&\quad + \|z\|^4 \Big(\frac{1}{12} \partial^4_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{6} \partial^3_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) \\
&\quad + \frac{1}{4} \partial^2_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0) + \frac{1}{6} \partial_{x_1} (\partial^\sharp_{y_1})^4 \varepsilon_d(0,0)\Big) + \|z\|^5 O(d^{-\beta}),
\end{aligned}$$

$$\begin{aligned}
\text{(B.27)}\quad &\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0) \\
&= \|z\|^2 \, \partial_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \|z\|^3 \left(\frac{1}{2} \partial^2_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{2} \partial_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0)\right) \\
&\quad + \|z\|^4 \left(\frac{1}{6} \partial^3_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{4} \partial^2_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) + \frac{1}{6} \partial_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0)\right) \\
&\quad + \|z\|^5 \Big(\frac{1}{24} \partial^4_{x_1} \partial^\sharp_{y_1} \varepsilon_d(0,0) + \frac{1}{12} \partial^3_{x_1} (\partial^\sharp_{y_1})^2 \varepsilon_d(0,0) \\
&\quad + \frac{1}{12} \partial^2_{x_1} (\partial^\sharp_{y_1})^3 \varepsilon_d(0,0) + \frac{1}{24} \partial_{x_1} (\partial^\sharp_{y_1})^4 \varepsilon_d(0,0)\Big) + \|z\|^6 O(d^{-\beta}).
\end{aligned}$$

Now, we can combine equations (B.24), (B.25), (B.26), and (B.27) with the expansions around 0 of $(b_+(\|z\|^2))^2$ (cf. equation (A.3)), $(b_-(\|z\|^2))^2$, and $b_+(\|z\|^2)b_-(\|z\|^2)$ (cf. equation (A.4)). Using Proposition 3.4 once again, we obtain

$$a_d^{(3)}(z) = \frac{1}{4v_2(\|z\|^2)}\|z\|^6 O(d^{-\beta}) = O((\ln d)^6 d^{-\beta}) = O(d^{-\alpha}),$$

where we used equations (A.2), (A.6), and the fact that $\|z\| \leqslant b_n \ln d$. This concludes the proof for $a_d^{(3)}$. As we already explained, we proceed similarly for the other nine coefficients to get (B.11), and the same kind of computations yield (B.12) and (B.13). ◻

Proof of Lemma 5.30. Let $\alpha \in (0,1)$, $x \in M$, and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We will denote by $L = (X, Y)$ a generic element of $\mathbb{R}^2 \otimes T_x^*M \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$. We also set $\chi(L) = |\det^\perp(X)|\,|\det^\perp(Y)|$. We have

$$\text{(B.28)} \qquad \mathbb{E}[|\det^\perp(X_d(z))|\,|\det^\perp(Y_d(z))|] =$$
$$\begin{aligned}
&= \frac{1}{(2\pi)^{rn}} \det(\Lambda_d(z))^{-1/2} \int \chi(L) \exp\left(-\frac{1}{2}\langle \Lambda_d(z)^{-1}L, L\rangle\right) \mathrm{d}L \\
&= \frac{1}{(2\pi)^{rn}} \left(\frac{\det \Lambda(z)}{\det \Lambda_d(z)}\right)^{1/2} \\
&\qquad \times \int \chi(\Lambda(z)^{1/2}L) \exp\left(-\frac{1}{2}\langle \Lambda(z)^{1/2}\Lambda_d(z)^{-1}\Lambda(z)^{1/2}L, L\rangle\right) \mathrm{d}L,
\end{aligned}$$

by a change of variable. By Lemma 5.29, we have

$$\det \Lambda_d(z) = (\det \Lambda(z))(1 + O(d^{-\alpha})).$$

If we set $\Xi_d(z) = \Lambda(z)^{1/2}\Lambda_d(z)^{-1}\Lambda(z)^{1/2} - \mathrm{Id}$, then $\Xi_d(z) = O(d^{-\alpha})$, and these estimates are uniform in (x, z). As in the proof of Lemma 4.18, by the Mean Value Theorem, for all L we have

$$\left|\exp\left(-\frac{1}{2}\langle \Xi_d(z)L, L\rangle\right) - 1\right| \leqslant \frac{1}{2}\|L\|^2 \|\Xi_d(z)\| \exp\left(\frac{1}{2}\|L\|^2 \|\Xi_d(z)\|\right).$$

Since $\Xi_d(z) = O(d^{-\alpha})$, for d large enough $\|\Xi_d(z)\| \leqslant \frac{1}{2}$. Hence,

$$\text{(B.29)} \qquad \int \chi(\Lambda(z)^{1/2}L) e^{-(1/2)\|L\|^2} \left|\exp\left(-\frac{1}{2}\langle \Xi_d(z)L, L\rangle\right) - 1\right| \mathrm{d}L$$
$$\leqslant \frac{\|\Xi_d(z)\|}{2} \int \chi(\Lambda(z)^{1/2}L)\|L\|^2 e^{-(1/4)\|L\|^2} \mathrm{d}L.$$

Recall that, by Lemma 4.12, the eigenvalues of the positive symmetric operator $\Lambda(z)$ are $u_1(\|z\|^2)$, $u_2(\|z\|^2)$, $1 + \exp(-\frac{1}{2}\|z\|^2)$, and $1 - \exp(-\frac{1}{2}\|z\|^2)$, with some multiplicities. These are bounded functions of z (see equations (A.2) and (A.5)). Hence, $\chi(\Lambda(z)^{1/2}L)$ is the square root of a polynomial in L whose coefficients are bounded functions of z. Thus, the integral on the righthand side of equation (B.29) is bounded, independently of (x,z). We get

$$\begin{aligned}\int \chi(\Lambda(z)^{1/2}L)\exp\left(-\frac{1}{2}\langle\Lambda(z)^{1/2}\Lambda_d(z)^{-1}\Lambda(z)^{1/2}L,L\rangle\right)\mathrm{d}L\\ = \int \chi(\Lambda(z)^{1/2}L)e^{-(1/2)\|L\|^2}\,\mathrm{d}L + O(d^{-\alpha})\\ = (2\pi)^{rn}\mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|] + O(d^{-\alpha}).\end{aligned}$$

Finally, by equation (B.28), we find

$$\begin{aligned}\mathbb{E}[|\det{}^{\perp}(X_d(z))|\,|\det{}^{\perp}(Y_d(z))|]\\ = \mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|] + O(d^{-\alpha}).\end{aligned} \tag{B.30}$$

By Lemma 4.13, for all $z \neq 0$, $\Lambda(z)$ is non-singular. Hence,

$$\mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|]$$

is a positive function of z. By Lemma 4.16,

$$\begin{aligned}\mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|]\\ = \mathbb{E}[|\det{}^{\perp}(X(\|z\|^2))|\,|\det{}^{\perp}(Y(\|z\|^2))|],\end{aligned}$$

and by Lemmas 4.17 and 4.18, if $r < n$, this quantity admits positive limits when $\|z\|$ goes to 0 or $\|z\|$ goes to $+\infty$. Thus, in this case,

$$\mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|]$$

is bounded from below by a positive constant, independent of (x,z). Then, equation (B.30) shows that

$$\begin{aligned}\mathbb{E}[|\det{}^{\perp}(X_d(z))|\,|\det{}^{\perp}(Y_d(z))|]\\ = \mathbb{E}[|\det{}^{\perp}(X_\infty(z))|\,|\det{}^{\perp}(Y_\infty(z))|](1 + O(d^{-\alpha})),\end{aligned}$$

and this concludes the proof for $r < n$.

If $r = n$, the leading term in equation (B.30) goes to 0 as $\|z\| \to 0$, so that we need to be more precise. From now on, we assume $r = n$. Let us assume for now that, in this case, we have

$$\int \chi(\Lambda(z)^{1/2}L)\|L\|^2 e^{-(1/4)\|L\|^2}\,\mathrm{d}L = O(\|z\|^2) \quad \text{as } z \to 0, \tag{B.31}$$

where the constant involved in the $O(\|z\|^2)$ is uniform in (x,z). Then, proceeding as we did in the case $r<n$, we get the following equivalent of equation (B.30):

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}(X_d(z))\right|\left|\det{}^{\perp}(Y_d(z))\right|\right]\\&\qquad=\mathbb{E}\left[\left|\det{}^{\perp}(X_\infty(z))\right|\left|\det{}^{\perp}(Y_\infty(z))\right|\right]+O(\|z\|^2d^{-\alpha}).\end{aligned}$$

By Lemma 4.17,

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}(X_\infty(z))\right|\left|\det{}^{\perp}(Y_\infty(z))\right|\right]\\&\qquad=\mathbb{E}\left[\left|\det{}^{\perp}(X(\|z\|^2))\right|\left|\det{}^{\perp}(Y(\|z\|^2))\right|\right]\sim\frac{n!}{2}\|z\|^2,\end{aligned}$$

as $z\to 0$. Hence,

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}(X_d(z))\right|\left|\det{}^{\perp}(Y_d(z))\right|\right]\\&\qquad=\mathbb{E}\left[\left|\det{}^{\perp}(X_\infty(z))\right|\left|\det{}^{\perp}(Y_\infty(z))\right|\right](1+O(d^{-\alpha}))\end{aligned}$$

uniformly for $x\in M$ and $\|z\|\leqslant 1$. In the domain $\|z\|\geqslant 1$,

$$\mathbb{E}\left[\left|\det{}^{\perp}(X_\infty(z))\right|\left|\det{}^{\perp}(Y_\infty(z))\right|\right]$$

is bounded from below by a positive constant independent of (x,z), and we proceed as in the case $r<n$, using equation (B.30). This yields the result for $r=n$.

To conclude the proof, we still have to prove that (B.31) holds when $r=n$. Let us write $L=(A,B)$ and $\Lambda(z)^{1/2}L=(X(z),Y(z))$ with A, B, $X(z)$, and $Y(z)\in T_x^*M\otimes\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. We choose any orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ and an orthonormal basis of T_xM such that the coordinates of z are $(\|z\|,0,\dots,0)$. We denote by (A_{ij}), (B_{ij}), $(X_{ij}(z))$, and $(Y_{ij}(z))\in\mathcal{M}_{rn}(\mathbb{R})$ the matrices of A, B, $X(z)$, and $Y(z)$ in these bases.

The matrix of $\Lambda(z)$ in the basis defined by $\mathcal{B}'_z$ (see Subsection 4.3) and our basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ is $\hat{\Lambda}(\|z\|^2)$, where $\hat{\Lambda}$ was defined by equation (A.15). That is, using the same notation as in the proof of Lemma 4.17 (see equations (A.9) and (A.10)), for all $i\in\{1,\dots,r\}$,

$$\begin{pmatrix}X_{i1}\\Y_{i1}\end{pmatrix}=\begin{pmatrix}\alpha(\|z\|^2)&\beta(\|z\|^2)\\\beta(\|z\|^2)&\alpha(\|z\|^2)\end{pmatrix}\begin{pmatrix}A_{i1}\\B_{i1}\end{pmatrix}$$

and

$$\begin{pmatrix}X_{ij}\\Y_{ij}\end{pmatrix}=\begin{pmatrix}\gamma(\|z\|^2)&\delta(\|z\|^2)\\\delta(\|z\|^2)&\gamma(\|z\|^2)\end{pmatrix}\begin{pmatrix}A_{ij}\\B_{ij}\end{pmatrix},\qquad\forall\, j\geqslant 2.$$

Hence, we have

$$\begin{aligned}\chi(\Lambda(z)^{1/2}L) &= \chi(X(z), Y(z)) = |\det{}^{\perp}(X(z))|\,|\det{}^{\perp}(Y(z))| \\ &= \Psi(\|z\|^2, (A_{ij}), (B_{ij})),\end{aligned}$$

where Ψ was defined by equation (A.11). Recall that Ψ satisfies (A.14) when $r = n$. As in the proof of Lemma 4.17 (cf. Appendix A), by Lebesgue's theorem we have

$$\begin{aligned}&\frac{2}{\|z\|^2}\int \chi(\Lambda(z)^{1/2}L)\|L\|^2 e^{-(1/4)\|L\|^2}\,\mathrm{d}L \\ &= \int \frac{2}{\|z\|^2}\Psi(\|z\|^2, (A_{ij}), (B_{ij}))\|L\|^2 e^{-(1/4)\|L\|^2}\,\mathrm{d}L \\ &\xrightarrow[\|z\|\to 0]{} \int \det\left(\frac{A_1 - B_1}{\sqrt{2}}, \frac{A_2 + B_2}{\sqrt{2}}, \dots, \frac{A_n + B_n}{\sqrt{2}}\right)^2 \|L\|^2 e^{-(1/4)\|L\|^2}\,\mathrm{d}L,\end{aligned} \tag{B.32}$$

where A_j (respectively, B_j) denotes the j-th column of the matrix of A (respectively, B) and $L = (A, B)$. This limit is finite, which proves that (B.31) is satisfied, and concludes the proof. ◻

Acknowledgements. Thomas Letendre was supported by the French National Research Agency through the ANR grants SpInQS (ANR-17-CE40-0011) and UniRaNDom (ANR-17-CE40-0008).

Martin Puchol was supported by the LABEX MILYON (ANR-10-LABX-0070) of the Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

We wish to thank Federico Dalmao and Maurizia Rossi for interesting discussions about Wiener-Itô expansions, and Xiaonan Ma for helpful discussions about Bergman kernels and for having encouraged our collaboration. Thomas Letendre also thanks Damien Gayet for suggesting this problem in the first place, and for his support.

REFERENCES

[1] R. J. ADLER and J. E. TAYLOR, *Random Fields and Geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR2319516)

[2] D. ARMENTANO, J. M. AZAÏS, F. DALMAO, and J. R. LEÓN, *Asymptotic variance of the number of real roots of random polynomial systems*, Proc. Amer. Math. Soc. **146** (2018), no. 12, 5437–5449. http://dx.doi.org/10.1090/proc/14215. MR3866880)

[3] J. M. AZAÏS and M. WSCHEBOR, *Level sets and extrema of random processes and fields*, John Wiley & Sons, Inc., Hoboken, NJ, 2009. http://dx.doi.org/10.1002/9780470434642. MR2478201)

[4] T. BAYRAKTAR, *Expected number of real roots for random linear combinations of orthogonal polynomials associated with radial weights*, Potential Anal. **48** (2018), no. 4, 459–471. http://dx.doi.org/10.1007/s11118-017-9643-9. MR3794381)

[5] R. J. BERMAN, *Bergman kernels and equilibrium measures for line bundles over projective manifolds*, Amer. J. Math. **131** (2009), no. 5, 1485–1524. http://dx.doi.org/10.1353/ajm.0.0077. MR2559862)

[6] P. BLEHER, B. SHIFFMAN, and S. ZELDITCH, *Universality and scaling of correlations between zeros on complex manifolds*, Invent. Math. **142** (2000), no. 2, 351–395. http://dx.doi.org/10.1007/s002220000092. MR1794066)

[7] E. BOGOMOLNY, O. BOHIGAS, and P. LEBOEUF, *Quantum chaotic dynamics and random polynomials*, J. Statist. Phys. **85** (1996), no. 5–6, 639–679. http://dx.doi.org/10.1007/BF02199359. MR1418808)

[8] Y. CANZANI and B. HANIN, *Local integral statistics for monochromatic random waves* (2016), available at http://arxiv.org/abs/arXiv:1610.09438.

[9] ———, *C^∞ scaling asymptotics for the spectral projector of the Laplacian*, J. Geom. Anal. **28** (2018), no. 1, 111–122. http://dx.doi.org/10.1007/s12220-017-9812-5. MR3745851)

[10] X. DAI, K. LIU, and X. MA, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), no. 1, 1–41. http://dx.doi.org/10.4310/jdg/1143593124. MR2215454)

[11] F. DALMAO, *Asymptotic variance and CLT for the number of zeros of Kostlan Shub Smale random polynomials*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 12, 1141–1145 (English, with English and French summaries). http://dx.doi.org/10.1016/j.crma.2015.09.016. MR3427922)

[12] F. DALMAO, I. NOURDIN, G. PECCATI, and M. ROSSI, *Phase singularities in complex arithmetic random waves* (2016), available at http://arxiv.org/abs/arXiv:1608.05631.

[13] D. GAYET and J. Y. WELSCHINGER, *Exponential rarefaction of real curves with many components*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), 69–96. http://dx.doi.org/10.1007/s10240-011-0033-3. MR2805598)

[14] ———, *Expected topology of random real algebraic submanifolds*, J. Inst. Math. Jussieu **14** (2015), no. 4, 673–702. http://dx.doi.org/10.1017/S1474748014000115. MR3394124)

[15] ———, *Betti numbers of random real hypersurfaces and determinants of random symmetric matrices*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 4, 733–772. http://dx.doi.org/10.4171/JEMS/601. MR3474455)

[16] Ph. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry*, Reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. http://dx.doi.org/10.1002/9781118032527. MR1288523)

[17] E. KOSTLAN, *On the distribution of roots of random polynomials*, From Topology to Computation, Berkeley, CA (1990), From Topology to Computation: Proceedings of the Smalefest, Springer, New York, 1993, pp. 419–431. MR1246137)

[18] M. F. KRATZ and J. R. LEÓN, *Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: Crossings and extremes*, Stochastic Process. Appl. **66** (1997), no. 2, 237–252. http://dx.doi.org/10.1016/S0304-4149(96)00122-6. MR1440400)

[19] ———, *Central limit theorems for level functionals of stationary Gaussian processes and fields*, J. Theoret. Probab. **14** (2001), no. 3, 639–672. http://dx.doi.org/10.1023/A:1017588905727. MR1860517)

[20] M. KRISHNAPUR, P. KURLBERG, and I. WIGMAN, *Nodal length fluctuations for arithmetic random waves*, Ann. of Math. (2) **177** (2013), no. 2, 699–737. http://dx.doi.org/10.4007/annals.2013.177.2.8. MR3010810)

[21] Th. LETENDRE, *Expected volume and Euler characteristic of random submanifolds*, J. Funct. Anal. **270** (2016), no. 8, 3047–3110. http://dx.doi.org/10.1016/j.jfa.2016.01.007. MR3470435)

[22] ———, *Variance of the volume of random real algebraic submanifolds*, Trans. Amer. Math. Soc. **371** (2019), no. 6, 4129–4192. http://dx.doi.org/10.1090/tran/7478. MR3917219)

[23] X. MA and G. MARINESCU, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007. MR2339952)

[24] ———, *Exponential estimate for the asymptotics of Bergman kernels*, Math. Ann. **362** (2015), no. 3–4, 1327–1347. http://dx.doi.org/10.1007/s00208-014-1137-0. MR3368102)

[25] D. MARINUCCI, G. PECCATI, M. ROSSI, and I. WIGMAN, *Non-universality of nodal length distribution for arithmetic random waves*, Geom. Funct. Anal. **26** (2016), no. 3, 926–960. http://dx.doi.org/10.1007/s00039-016-0376-5. MR3540457)

[26] F. NAZAROV and M. SODIN, *Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions*, Zh. Mat. Fiz. Anal. Geom. **12** (2016), no. 3, 205–278. http://dx.doi.org/10.15407/mag12.03.205. MR3522141)

[27] L. I. NICOLAESCU, *Critical sets of random smooth functions on compact manifolds*, Asian J. Math. **19** (2015), no. 3, 391–432. http://dx.doi.org/10.4310/AJM.2015.v19.n3.a2. MR3361277)

[28] I. NOURDIN and G. PECCATI, *Normal Approximations with Malliavin Calculus*: *From Stein's Method to Universality*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012. http://dx.doi.org/10.1017/CBO9781139084659. MR2962301)

[29] P. SARNAK and I. WIGMAN, *Topologies of nodal sets of random band limited functions*, Advances in the Theory of Automorphic Forms and their L-functions, Contemp. Math., vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 351–365. http://dx.doi.org/10.1090/conm/664/13040. MR3502990)

[30] B. SHIFFMAN and S. ZELDITCH, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), no. 3, 661–683. http://dx.doi.org/10.1007/s002200050544. MR1675133)

[31] ______, *Number variance of random zeros on complex manifolds, II*: *Smooth statistics*, Pure Appl. Math. Q. **6** (2010), no. 4, Special Issue: In honor of Joseph J. Kohn, 1145–1167. http://dx.doi.org/10.4310/PAMQ.2010.v6.n4.a10. MR2742043)

[32] R. SILHOL, *Real Algebraic Surfaces*, Lecture Notes in Mathematics, vol. 1392, Springer-Verlag, Berlin, 1989. http://dx.doi.org/10.1007/BFb0088815. MR1015720)

[33] E. SLUD, *Multiple Wiener-Itô integral expansions for level-crossing-count functionals*, Probab. Theory Related Fields **87** (1991), no. 3, 349–364. http://dx.doi.org/10.1007/BF01312215. MR1084335)

[34] T. TAO and V. VU, *Local universality of zeroes of random polynomials*, Int. Math. Res. Not. IMRN **13** (2015), 5053–5139. http://dx.doi.org/10.1093/imrn/rnu084. MR3439098)

[35] I. WIGMAN, *Fluctuations of the nodal length of random spherical harmonics*, Comm. Math. Phys. **298** (2010), no. 3, 787–831. http://dx.doi.org/10.1007/s00220-010-1078-8. MR2670928)

THOMAS LETENDRE:
CNRS, Sorbonne Université
Institut de Mathématiques de Jussieu-Paris Rive Gauche
F-75005 Paris
France
E-MAIL: letendre@math.cnrs.fr

MARTIN PUCHOL:
Université Paris-Sud
Laboratoire de Mathématiques d'Orsay
F-91405 Orsay
France
E-MAIL: martin.puchol@math.cnrs.fr

KEY WORDS AND PHRASES: Random submanifolds, Kac-Rice formula, linear statistics, Kostlan-Shub-Smale polynomials, Bergman kernel, real projective manifold, Wiener-Ito expansion.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 14P99, 32A25, 53C40, 60G15, 60G57.

Received: September 6, 2017.